

# Transverse Equilibrium Distribution Functions

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**S1: Transverse Vlasov-Poisson Model:** for a coasting, single species beam with electrostatic self-fields propagating in a linear focusing lattice:

$\mathbf{x}_\perp, \mathbf{x}'_\perp$  transverse particle coordinate, angle  
 $f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$  single particle distribution

$q, m$  charge, mass  
 $\gamma_b, \beta_b$  axial relativistic factors  
 $H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$  single particle Hamiltonian

**Vlasov Equation** (see J.J. Barnard, [Introductory Lectures](#)):

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

**Particle Equations of Motion:**

$$\frac{d}{ds} \mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \quad \frac{d}{ds} \mathbf{x}'_\perp = - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

**Hamiltonian** (see S.M. Lund, lectures on [Transverse Particle Equations of Motion](#)):

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} \kappa_x(s) x^2 + \frac{1}{2} \kappa_y(s) y^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

**Poisson Equation:**

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = - \frac{q}{\epsilon_0} \int d^2 \mathbf{x}'_\perp f_\perp$$

+ boundary conditions on  $\phi$

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**Hamiltonian expression of the Vlasov equation:**

$$\begin{aligned} \frac{d}{ds} f_\perp &= \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0 \\ &= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0 \end{aligned}$$

**Using the equations of motion:**

$$\begin{aligned} \frac{d}{ds} \mathbf{x}_\perp &= \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} = \mathbf{x}'_\perp \\ \frac{d}{ds} \mathbf{x}'_\perp &= - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = - \left( \kappa_x x \hat{\mathbf{x}} + \kappa_y y \hat{\mathbf{y}} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial \mathbf{x}_\perp} \right) \end{aligned}$$

In formal dynamics, a "Poisson Bracket" notation is frequently employed:

$$\begin{aligned} \frac{d}{ds} f_\perp &= \frac{\partial f_\perp}{\partial s} + \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} \cdot \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} - \left( \kappa_x x \hat{\mathbf{x}} + \kappa_y y \hat{\mathbf{y}} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0 \end{aligned}$$

$$\frac{\partial f_\perp}{\partial s} + \mathbf{x}'_\perp \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left( \kappa_x x \hat{\mathbf{x}} + \kappa_y y \hat{\mathbf{y}} + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial \mathbf{x}_\perp} \right) \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

**Poisson Bracket**

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## Comments on Vlasov-Poisson Model

- ♦ Collisionless Vlasov-Poisson model good for intense beams with many particles
  - Collisions negligible, see: J.J. Barnard, [Intro. Lectures](#)
- ♦ Vlasov-Poisson model can be solved as an initial value problem
  - 1)  $f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s = s_i) =$  Initial "condition" (function) specified
  - 2) Vlasov-Poisson model solved for subsequent evolution in  $s$  for  $f_{\perp}(\mathbf{x}_{\perp}, \mathbf{x}'_{\perp}, s)$  for  $s \geq s_i$

♦ The coupling to the self-field via the Poisson equation makes the Vlasov-Poisson model *highly nonlinear*

$$\rho = q \int d^2x'_{\perp} f_{\perp} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = - \frac{\rho}{\epsilon_0}$$

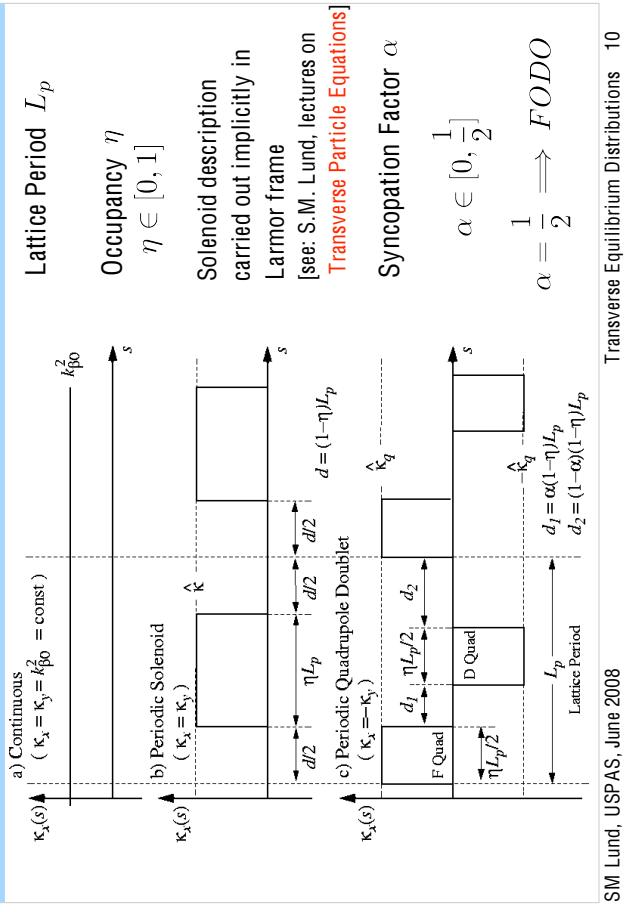
♦ Vlasov-Poisson system is written without acceleration, but the transforms developed to identify the normalized emittance in the lectures on

[Transverse Particle Equations of Motion](#) can be exploited to generalize all result presented to (weakly) accelerating beams (interpret in tilde variables)

♦ For solenoidal focusing the system must be interpreted in the rotating Larmor Frame, see: [lectures on Transverse Particle Equations of Motion](#)

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## Review: Focusing lattices, continuous and periodic (simple piecewise constant):



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## Example Hamiltonians:

Continuous focusing  $K_x = K_y = k_{\beta 0}^2 = \text{const}$

$$H_{\perp} = \frac{1}{2} \mathbf{x}'_{\perp}^2 + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_{\perp}^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Solenoidal focusing (in Larmor frame variables)  $K_x = K_y = \kappa(s)$

$$H_{\perp} = \frac{1}{2} \mathbf{x}'_{\perp}^2 + \frac{1}{2} \kappa \mathbf{x}_{\perp}^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Quadrupole focusing  $\kappa_x = -\kappa_y = \kappa(s)$

$$H_{\perp} = \frac{1}{2} \mathbf{x}'_{\perp}^2 + \frac{1}{2} \kappa x^2 - \frac{1}{2} \kappa y^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi$$

Review: Undepressed particle phase advance  $\alpha_0$  is typically employed to characterize the applied focusing strength of periodic lattices:

see: S.M. Lund lectures on [Transverse Particle Equations of Motion](#)  
x-orbit without space-charge satisfies Hill's equation

$$x''(s) + \kappa_x(s)x(s) = 0$$

$$\left( \begin{array}{c} x(s) \\ x'(s) \end{array} \right) = \mathbf{M}_x(s \mid s_i) \cdot \left( \begin{array}{c} x(s_i) \\ x'(s_i) \end{array} \right)$$

Undepressed phase advance

$$\cos \sigma_{0x} = \frac{1}{2} \text{Tr } \mathbf{M}_x(s_i + L_p \mid s_i)$$

♦ Subscript used stresses -plane value and zero ( $= 0$ ) space-charge effects  
Single particle (and centroid) stability requires:  
 $\frac{1}{2} \text{Tr } \mathbf{M}_x(s_i + L_p \mid s_i) < 1 \rightarrow \sigma_{0x} < 180^\circ$

Analogous equations hold in the y-plane

[Courant and Snyder, Annals of Phys. 3, 1 (1958)]

The **undepressed phase advance** can also be equivalently calculated from:

$$w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \quad w_{0x}(s + L_p) = w_{0x}(s)$$

$$\sigma_{0x} = \int_{s_i}^{s_i + L_p} \frac{ds}{w_{0x}^2} \quad w_{0x} > 0$$

♦ Subscript **stresses** - plane value and zero ( $= 0$ ) space-charge effects

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/// Example: Continuous focusing  $f_\perp = f_\perp(H_\perp)$

$$H_\perp = \frac{1}{2} \mathbf{x}'_\perp^2 + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi \quad \text{no explicit dependence}$$

$$\frac{df_\perp}{ds} = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial s} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \quad \text{see problem sets for detailed argument}$$

$$= \frac{\partial f_\perp}{\partial H_\perp} \frac{\partial H_\perp}{\partial s} + \frac{\partial f_\perp}{\partial H_\perp} \left( \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \cdot \frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \right) = 0$$

Showing that  $f_\perp = f_\perp(H_\perp)$  exactly satisfies Vlasov's equation for continuous focusing

♦ Also, for physical solutions must require:  $f_\perp(H_\perp) \geq 0$

- To be appropriate for single species with positive density

♦ Huge variety of equilibrium function choices  $f_\perp(H_\perp)$  can be made to generate many radically different equilibria

- Infinite variety in function space

♦ Does *NOT* apply to systems with -varying focusing  $\kappa_x \rightarrow k_{\beta b}^2$

- Can provide a rough guide if we can approximate:

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**S2: Vlasov Equilibria:** Plasma physics-like approach is to resolve the system into an equilibrium + perturbation and analyze stability

Equilibrium constructed from single-particle constants of motion  $C_i$

$$f_\perp = f_\perp(\{C_i\}) \geq 0 \quad \Rightarrow \quad \text{equilibrium}$$

$$\frac{d}{ds} f_\perp(\{C_i\}) = \sum_i \frac{\partial f_\perp}{\partial C_i} \frac{dC_i}{ds} = 0$$

Comments:

- ♦ **Equilibrium** is an exact solution to Vlasov's equation that *does not change* in 4D phase-space as advances
  - **Projections** of the distribution can evolve in in general cases
  - ♦  $f_\perp(\{C_i\})$  results from single particle species
  - ♦ Particle conservation constraints are in the presence of (possibly -varying) applied and space-charge forces
  - Highly non-trivial!
  - Only one exact solution known for -varying focusing using Courant-Snyder invariants: the KV distribution to be analyzed in this lecture

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Typical single particle constants of motion:

Transverse Hamiltonian for continuous focusing:

$$H_\perp = \frac{1}{2} \mathbf{x}_\perp^2 + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m \gamma_b^3 \beta_b^2 c^2} \phi = \text{const}$$

$$k_{\beta 0}^2 = \text{const}$$

- ♦ Not valid for periodic focusing systems!

Angular momentum for systems invariant under azimuthal rotation:

$$P_\theta = xy' - yx' = \text{const}$$

- ♦ Subtle point: This form is really a **Canonical Angular Momentum** and applies to solenoidal magnetic focusing when the variables are expressed in the **rotating Larmor frame** (i.e., in the  $\tilde{x}$  variables)
- see: S.M. Lund, lectures on **Transverse Particle Equations**

Axial kinetic energy for systems with no acceleration:

$$\mathcal{E} = (\gamma_b - 1)mc^2 = \text{const}$$

- ♦ Trivial for a coasting beam with  $\gamma_b \beta_b = \text{const}$

More on other classes of constraints later ...

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## Plasma physics approach to beam physics:

**Resolve:**  $f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$

equilibrium perturbation  $f_\perp \gg |\delta f_\perp|$

and carry out equilibrium + stability analysis

**Comments:**

- Attraction is to parallel the impressive successes of plasma physics
- Gain insight into preferred state of nature

- Beams are born off a source and may not be close to an equilibrium condition
- Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the (unphysical) KV distribution
- Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations
- It is not clear if smooth Vlasov equilibria exist in periodic focusing
- Higher model detail vastly complicates picture!
- If system can be tuned to more closely resemble a relaxed, equilibrium, one might expect less deleterious effects based on plasma physics analogies

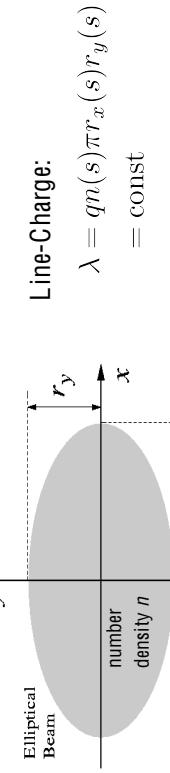
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## S3: The KV Equilibrium Distribution

[Kapchinskij and Vladimirkij, Proc. Int. Conf. On High Energy Accel., p. 274 (1959); and Review: Lund, Kikuchi, and Davidson, PRSTAB, to be published, (2008)]

Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:  
 $\lambda = qm(s)\pi r_x(s)r_y(s)$   
 $= \text{const}$

Free-space self-field solution within the beam (see: [Appendix A](#)) is:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0} \left[ \frac{x^2}{(r_x + r_y)r_x} + \frac{y^2}{(r_x + r_y)r_y} \right] + \text{const}$$

$$\begin{aligned} -\frac{\partial\phi}{\partial x} &= \frac{\lambda}{\pi\epsilon_0} \frac{x}{(r_x + r_y)r_x} \\ -\frac{\partial\phi}{\partial y} &= \frac{\lambda}{\pi\epsilon_0} \frac{y}{(r_x + r_y)r_y} \end{aligned}$$

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If we regard the envelope radii  $r_x$ ,  $r_y$  as specified functions of  $s$ , then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$\begin{aligned} x''(s) + \kappa_x^{\text{eff}}(s)x(s) &= 0 \\ y''(s) + \kappa_y^{\text{eff}}(s)y(s) &= 0 \\ \kappa_x^{\text{eff}}(s) &= \kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)} \\ \kappa_y^{\text{eff}}(s) &= \kappa_y(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_y(s)} \end{aligned}$$

**Suggests Procedure:**

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
- Nontrivial step:** guess and show that it works

- Resulting distribution will be an equilibrium that does not evolve in 4D phase-space, but lower-dimensional phase-space projections can evolve in
- Same measure of space-charge intensity used by J.J. Barnard in [Intro. Lectures](#)
  - Properties/interpretations of the pervenue will be extensively developed in this and subsequent lectures

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## Review (1): The Courant-Snyder invariant of Hills equation

[Courant and Snyder, Annl. Phys. 3, 1 (1958)]

Hills equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$x''(s) + \kappa(s)x(s) = 0$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s) \cos \psi(s)$$

where  $w(s)$  is the periodic amplitude function satisfying

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \quad w(s) > 0$$

$\psi(s)$  is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^s \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$A_i$  and  $\psi_i$  are constants set by initial conditions at  $s = s_i$

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Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of -orbit equations:

$$x(s) = A_{xi} w_x(s) \cos \psi_x(s)$$

$$x'(s) = A_{xi} w'_x(s) \cos \psi_x(s) - \frac{A_{xi}}{w_x(s)} \sin \psi_x(s) = \text{const}$$

where

$$w_x''(s) + \kappa_x(s)w_x(s) - \frac{2Q}{[r_x(s) + r_y(s)][r_x(s)w_x(s) - w_x^3(s)]} = 0$$

$$w_x(s + L_p) = w_x(s) \quad w_x(s) > 0$$

$$\psi_x(s) = \psi_{xi} + \int_{s_i}^s \frac{d\tilde{s}}{w_x^2(\tilde{s})}$$

identifies the Courant-Snyder invariant

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w'_x x)^2 = A_{xi}^2 = \text{const}$$

Analogous equations hold for the  $y$ -plane

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The KV envelope equations:

Define maximum Courant-Snyder invariants:

$$\varepsilon_x \equiv \text{Max}(A_{xi}^2)$$

$$\varepsilon_y \equiv \text{Max}(A_{yi}^2)$$

These values must correspond to the beam-edge:

$$r_x(s) = \sqrt{\varepsilon_x} w_x(s)$$

$$r_y(s) = \sqrt{\varepsilon_y} w_y(s)$$

The equations for  $w_x$  and  $w_y$  can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$r_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$

$$r_y''(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$

$$r_x(s + L_p) = r_x(s) \quad r_x(s) > 0$$

$$r_y(s + L_p) = r_y(s) \quad r_y(s) > 0$$

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Review (2): The Courant-Snyder invariant of Hills equation

From this formulation, it follows that

$$x(s) = A_i w(s) \cos \psi(s)$$

$$x'(s) = A_i w'(s) \cos \psi(s) - \frac{A_i}{w(s)} \sin \psi(s)$$

or

$$\frac{x}{w} = A_i \cos \psi$$

$$wx' - w'x = A_i \sin \psi$$

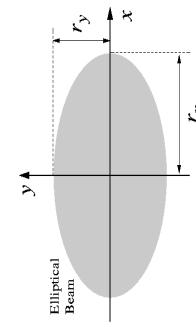
square and add equations to obtain the Courant-Snyder invariant

$$\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}$$

- ♦ Simplifies interpretation of dynamics
- ♦ Extensively used in accelerator physics

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Use variable rescalings to denote - and -plane Courant-Snyder invariants as:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w'_x x)^2 = A_{xi}^2 = \text{const}$$

$$\begin{aligned} \left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 &= C_x = \text{const} \\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 &= C_y = \text{const} \end{aligned}$$

**Kapchinskij and Vladimirov** constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta[C_x + C_y - 1]$$

- ♦ Delta function means the sum of the x- and y-invariants is a constant
- ♦ Other forms cannot generate the needed uniform density elliptical beam projection (see: [S9](#))

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The KV equilibrium is constructed from the Courant-Snyder invariants:

**KV equilibrium distribution:**

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta \left[ \left( \frac{x}{r_x} \right)^2 + \left( \frac{r_x x' - r'_x x}{\varepsilon_x} \right)^2 + \left( \frac{y}{r_y} \right)^2 + \left( \frac{r_y y' - r'_y y}{\varepsilon_y} \right)^2 - 1 \right] = \text{const}$$

$\delta(x)$  = Dirac delta function

This distribution generates (see: proof in [Appendix B](#)) the correct uniform density elliptical beam:

$$n = \int d^2 x'_\perp f_\perp = \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Obtaining this form consistent with the assumptions, thereby

**demonstrating full self-consistency of the KV equilibrium distribution.**

- Full 4-D form of the distribution does not evolve in
- Projections of the distribution can (and generally do!) evolve in

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**/// Comment on notation of integrals:**

- 2<sup>nd</sup> forms useful for systems with azimuthal spatial or annular symmetry

Spatial

$$\int d^2 x_\perp \dots \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \dots$$

$$\begin{aligned} &= \int_0^{\infty} dr r \int_{-\pi}^{\pi} d\theta \dots && \text{Cylindrical Coordinates:} \\ &\quad x = r \cos \theta && x' \in (-\infty, \infty) \\ &\quad y = r \sin \theta && y' \in (-\infty, \infty) \end{aligned}$$

Angular

$$\begin{aligned} \int d^2 x'_\perp \dots &\equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \dots && \text{Angular} \\ &= \int_0^{\infty} dr' r' \int_{-\pi}^{\pi} d\theta' \dots && \text{Cylindrical Coordinates:} \\ &\quad x' = r' \cos \theta' && x' \in (-\infty, \infty) \\ &\quad y' = r' \sin \theta' && y' \in (-\infty, \infty) \end{aligned}$$

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**Comment on notation of integrals (continued):**  
Axysymmetry simplifications

Spatial: for some function  $f(\mathbf{x}_\perp^2) = f(r^2)$

$$\begin{aligned} \int d^2x_\perp f(\mathbf{x}_\perp^2) &= 2\pi \int_0^\infty dr r f(r^2) \\ &= \pi \int_0^\infty dr'^2 f(r'^2) \\ &= \pi \int_0^\infty dw f(w) \end{aligned}$$

Angular: for some function  $g(\mathbf{x}_\perp^2) = g(r'^2)$

$$\begin{aligned} \int d^2x'_\perp g(\mathbf{x}'_\perp^2) &= 2\pi \int_0^\infty dr' r' g(r'^2) \\ &= \pi \int_0^\infty dr'^2 g(r'^2) \\ &= \pi \int_0^\infty du g(u) \end{aligned}$$

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Transverse Equilibrium Distributions 29  
*//*

Moments of the KV distribution can be calculated directly from the distribution to further aid interpretation: [see: [Appendix B](#) for details]

Full 4D average:	$\langle \dots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \dots f_\perp}{\int d^2x_\perp \int d^2x'_\perp f_\perp}$
Restricted angle average:	$\langle \dots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \dots f_\perp}{\int d^2x'_\perp f_\perp}$

Envelope edge radius:

$$r_x = 2\langle x'^2 \rangle_\perp^{1/2}$$

rms edge emittance (maximum Courant-Snyder invariant):

$$\varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]^{1/2} = \text{const}$$

Coherent flows (within the beam, zero otherwise):

$$\langle x' \rangle_{\mathbf{x}'_\perp} = r'_x \frac{x}{r'_x}$$

Angular spread (x-temperature, within the beam, zero otherwise):

$$T_x \equiv \langle (x' - \langle x' \rangle_{\mathbf{x}'_\perp})^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2} \left( 1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} \right)$$

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Transverse Equilibrium Distributions 30

**Summary of 1<sup>st</sup> and 2<sup>nd</sup> order moments of the KV distribution:**

Canonical transformation illustrates KV distribution structure:  
[Davidson Physics of Nonneutral Plasmas, Addison-Wesley (1990), and [Appendix B](#)]  
Phase-space transformation:

$$\begin{aligned} X &= \frac{\sqrt{\varepsilon_x}}{r_x} x \\ X' &= \frac{r'_x x' - r'_x x}{\sqrt{\varepsilon_x}} \end{aligned}$$

Courant-Snyder invariants in the presence of beam space-charge are then simply:

$$X^2 + X'^2 = \text{const}$$

and the KV distribution takes the simple, symmetrical form:

$$f_\perp(x, y, x', y', s) = f_\perp(X, Y, X', Y') = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta \left[ \frac{X^2 + X'^2}{\varepsilon_x} + \frac{Y^2 + Y'^2}{\varepsilon_y} - 1 \right]$$

from which the density and other projections can be (see: [Appendix B](#)) calculated more easily:  $n = \int d^2x'_\perp f_\perp = \frac{\lambda}{q\pi r_x r_y} \int_0^\infty dU^2 \delta \left[ U^2 - \left( 1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} \right) \right]$

$$= \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

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Transverse Equilibrium Distributions 31

**Summary of 1<sup>st</sup> and 2<sup>nd</sup> order moments of the KV distribution:**

Moment	Value	All 1 <sup>st</sup> and 2 <sup>nd</sup> order moments not listed vanish, i.e.,
$\int d^2x'_\perp x' f_\perp$	$r'_x \frac{x}{r_x} n$	$\int d^2x'_\perp xy f_\perp = 0$
$\int d^2x'_\perp y' f_\perp$	$r'_y \frac{y}{r_y} n$	$\langle xy \rangle_\perp = 0$
$\int d^2x'_\perp x' y' f_\perp$	$0$	see reviews by:
$\langle x^2 \rangle_\perp$	$\frac{r_x^2}{4} n$	(limit of results presented)
$\langle x'^2 \rangle_\perp$	$\frac{r_x'^2}{4} + \frac{\varepsilon_x^2}{4r_x^2} n$	Lund and Bukh, PRSTAB 024801 (2004), Appendix A
$\langle y^2 \rangle_\perp$	$\frac{r_y^2}{4} + \frac{\varepsilon_y^2}{4r_y^2} n$	S.M. Lund, T. Kikuchi, and R.C. Davidson, PRSTAB, to be published (2008)
$\langle xx' \rangle_\perp$	$\frac{r_x r'_x}{4} n$	
$\langle yy' \rangle_\perp$	$\frac{r_y r'_y}{4} n$	
$\frac{\langle xy' - yx' \rangle_\perp}{16[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]}$	$\frac{\varepsilon_x^2}{\varepsilon_y^2} n$	
$\frac{\langle yy' \rangle_\perp}{16[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp^2]}$	$\frac{\varepsilon_y^2}{\varepsilon_x^2} n$	

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Transverse Equilibrium Distributions 32

## KV Envelope equation

The envelope equation reflects low-order force balances

$$\begin{aligned} r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} &= 0 & \text{Matched Solution:} \\ r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} &= 0 & r_x(s + L_p) = r_x(s) \\ \varepsilon_x &= \varepsilon_y & r_y(s + L_p) = r_y(s) \end{aligned}$$

Comments:

- ♦ Envelope equation is a projection of the 4D invariant distribution
  - Envelope evolution equivalently given by moments of the 4D equilibrium distribution

- ♦ **Most important basic design equation** for transport lattices with high space-charge intensity
  - Simplest consistent model incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design!

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Transverse Equilibrium Distributions 33

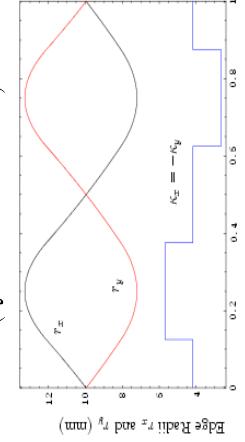
The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

<b>Parameters</b>	$L_p = 0.5 \text{ m}$ , $\sigma_0 = 80^\circ$ , $\eta = 0.5$
	$\varepsilon_x = 50 \text{ mm-mrad}$
	$\sigma/\sigma_0 = 0.2$

$$\begin{aligned} r_x(s + L_p) &= r_x(s) \\ r_y(s + L_p) &= r_y(s) \\ \varepsilon_x &= \varepsilon_y \end{aligned}$$

$$(Q = 6.6986 \times 10^{-4})$$

$$(Q = 6.5614 \times 10^{-4})$$



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

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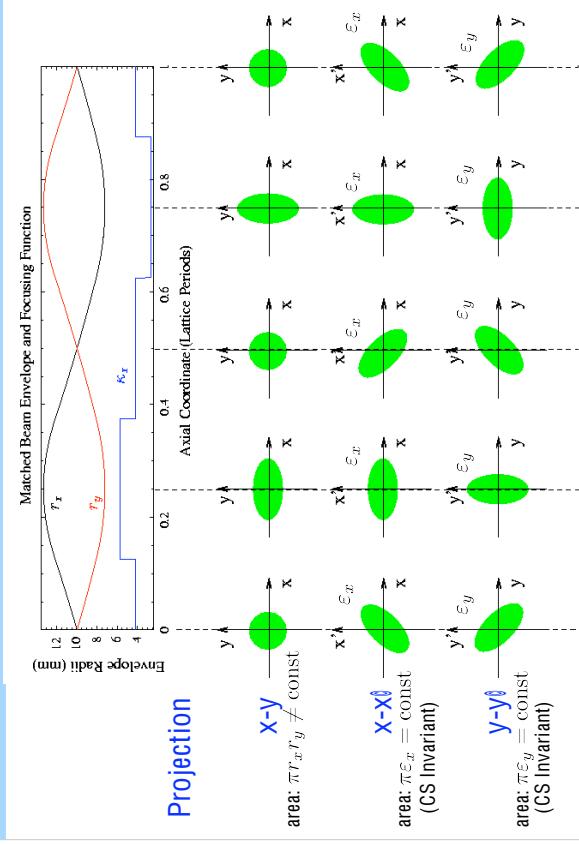
Transverse Equilibrium Distributions 35

## Comments Continued:

- ♦ Beam envelope matching will be covered in much more detail in S.M. Lund lectures on **Centroid and Envelope Description of Beams**
  - Requires specific initial conditions for periodic evolution
- $r_x(s_i)$ ,  $r_y(s_i)$
- ♦ Instabilities of envelope equations are well understood and real (to be covered: see S.M. Lund lectures on **Centroid and Envelope Description of Beams**)
  - Must be avoided for reliable machine operation

Transverse Equilibrium Distributions 34

Some phase-space projections of a matched KV equilibrium beam in a periodic FODO quadrupole transport lattice



Transverse Equilibrium Distributions 36

KV model shows that particle orbits in the presence of space-charge can be strongly modified  $\pm$  space charge slows the orbit response:

Matched envelope:

$$\begin{aligned} r_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} &= 0 \\ r_x(s + L_p) &= r_x(s) & r_x(s) > 0 \\ r_y(s + L_p) &= r_y(s) & r_y(s) > 0 \end{aligned}$$

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

$$x''(s) + \kappa_x(s)x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)}x(s) = 0$$

All particles have the same value of depressed phase advance (similar Eqns in):

$$\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_x^2(s)}$$

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Transverse Equilibrium Distributions 37

Contrast: Review, the undepressed particle phase advance calculated in the lectures on **Transverse Particle Equations of Motion**

The undepressed phase advance is defined as the phase advance of a particle in the absence of space-charge ( $\varepsilon = 0$ ):

- Denote by  $\sigma_{0x}$  to distinguish from the "depressed" phase advance  $\sigma_x$  in the presence of space-charge

$$\begin{aligned} w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} &= 0 & w_{0x}(s + L_p) &= w_{0x}(s) \\ w_{0x} > 0 \\ \sigma_{0x} &= \int_{s_i}^{s_i + L_p} \frac{ds}{w_{0x}^2} \end{aligned}$$

This can be equivalently calculated from the matched envelope with  $\varepsilon = 0$ :

$$\begin{aligned} r_{0x}'' + \kappa_x r_{0x} - \frac{\varepsilon_x^2}{r_{0x}^3} &= 0 & r_{0x}(s + L_p) &= r_{0x}(s) \\ r_{0x} > 0 \\ \sigma_{0x} &= \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_{0x}^2} \end{aligned}$$

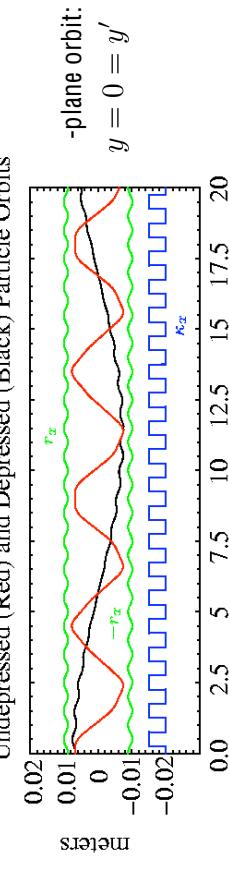
- Value of  $\varepsilon_x$  is arbitrary (answer for  $\sigma_{0x}$ 's independent)

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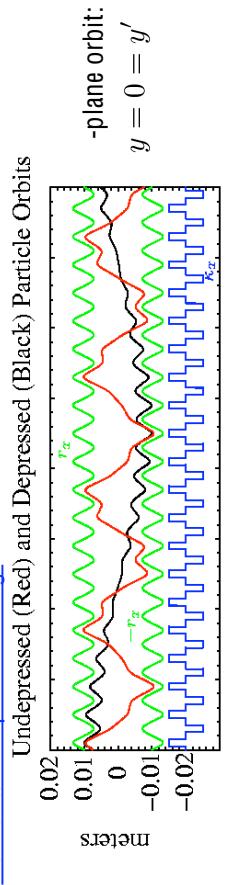
**Depressed particle -plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown**

Solenoidal Focusing (Larmor frame orbit):

Undepressed (Red) and Depressed (Black) Particle Orbits



FODO Quadrupole Focusing: Undepressed (Red) and Depressed (Black) Particle Orbits



Contrast: Review, the undepressed particle phase advance calculated in the lectures on **Transverse Particle Equations of Motion**

The undepressed phase advance is defined as the phase advance of a particle in the absence of space-charge ( $\varepsilon = 0$ ):

- Denote by  $\sigma_{0x}$  to distinguish from the "depressed" phase advance  $\sigma_x$  in the presence of space-charge

$$\begin{aligned} w_{0x}'' + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} &= 0 & w_{0x}(s + L_p) &= w_{0x}(s) \\ w_{0x} > 0 \\ \sigma_{0x} &= \int_{s_i}^{s_i + L_p} \frac{ds}{w_{0x}^2} \end{aligned}$$

This can be equivalently calculated from the matched envelope with  $\varepsilon = 0$ :

$$\begin{aligned} r_{0x}'' + \kappa_x r_{0x} - \frac{\varepsilon_x^2}{r_{0x}^3} &= 0 & r_{0x}(s + L_p) &= r_{0x}(s) \\ r_{0x} > 0 \\ \sigma_{0x} &= \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_{0x}^2} \end{aligned}$$

- Value of  $\varepsilon_x$  is arbitrary (answer for  $\sigma_{0x}$ 's independent)

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**Depressed particle phase advance provides a convenient measure of space-charge strength**

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0$$

**Depressed phase advance** within a matched beam

$$\sigma = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{r_y^2(s)}$$

$$\lim_{Q \rightarrow 0} \sigma = \sigma_0$$

Normalized space charge strength	$\sigma/\sigma_0 \rightarrow 0$	Cold Beam (space-charge dominated) $\varepsilon \rightarrow 0$
$0 \leq \sigma/\sigma_0 \leq 1$	$\sigma/\sigma_0 \rightarrow 1$	Warm Beam (kinetic dominated) $Q \rightarrow 0$

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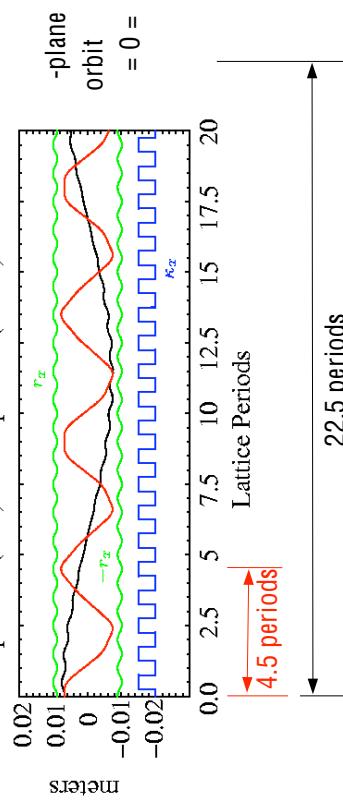
Transverse Equilibrium Distributions 40

For example matched envelope presented earlier:

**Undepressed phase advance:**  $\sigma_0 = 80^\circ$   
Depressed phase advance:  $\sigma = 16^\circ \rightarrow \sigma/\sigma_0 = 0.2$

**Solenoidal Focusing** (Larmor frame orbit):

Undepressed (Red) and Depressed (Black) Particle Orbits



**Comment:**

All particles in the distribution will, of course, always move in response to both applied and self-fields. You cannot turn off space-charge for an undepressed orbit. It is a convenient conceptual construction to help understand focusing properties.

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repeat periods  
**4.5**

phase advance:  $\sigma = 16^\circ \rightarrow \sigma/\sigma_0 = 0.2$   
22.5  
Periods for 360 degree phase advance

The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution

For the same focusing lattice, replace any beam charge  $\rho(x, y)$  density by a uniform density KV beam in each axial slice ( $s$ ) using averages calculated from the actual "real" beam distribution with:

$$\langle \dots \rangle_{\perp} \equiv \frac{\int d^2x_{\perp} \int d^2x'_{\perp} \dots f_{\perp}}{\int d^2x_{\perp} \int d^2x'_{\perp} f_{\perp}} \quad f_{\perp} = \text{real distribution}$$

rms equivalent beam:

Quantity	KV Equiv.	Calculated from Distribution
Pervance	$Q$	$= q^2 \int d^2x_{\perp} \int d^2x'_{\perp} f_{\perp} / [2\pi\epsilon_0\gamma_b^3\beta_b^2 c^2]$
x-edge radius	$r_x$	$= 2\langle x^2 \rangle_{\perp}^{1/2}$
y-edge radius	$r_y$	$= 2\langle y^2 \rangle_{\perp}^{1/2}$
x-emittance	$\varepsilon_x$	$= 4[\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}]^{1/2}$
y-emittance	$\varepsilon_y$	$= 4[\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}]^{1/2}$

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**Comments on rms equivalent beam concept:**

- ◆ The emittances will generally evolve in
  - Means that the equivalence must be recalculated in every slice as the emittances evolve
  - For reasons to be analyzed later (see S.M. Lund lectures on **Kinetic Stability of Beams**), this evolution is often small

◆ Concept is highly useful

- KV equilibrium properties well understood and are approximately correct to model lowest order "real" beam properties
- See, Reiser, *Theory and Design of Charged Particle Beams* (1994, 2008) for a detailed discussion of rms equivalence

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \quad \lambda = q \int d^2x_{\perp} \rho = \text{const}$$

$$r_x = 2\langle x^2 \rangle_{\perp}^{1/2} \quad \varepsilon_x = 4[\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2]^{1/2}$$

$$r_y = 2\langle y^2 \rangle_{\perp}^{1/2} \quad \varepsilon_y = 4[\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}^2]^{1/2}$$

The emittances must, in general, evolve in  $s$  under this model  
(see SM Lund lectures on *Transverse Kinetic Stability*)

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Transverse Equilibrium Distributions 44

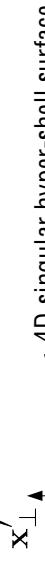
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## Further comments on the KV equilibrium: Distribution Structure

### KV equilibrium distribution:

$$f_{\perp} \sim \delta[\text{Courant-Snyder invariants}]$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:  


- ◆ Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see: lectures by S.M. Lund on [Centroid and Envelope Descriptions of Beams](#))

- ◆ Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. 13, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see: lectures by S.M. Lund on [Kinetic Stability of Beams](#))
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

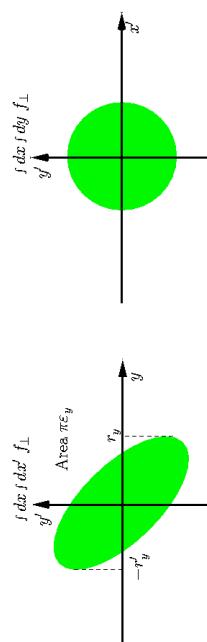
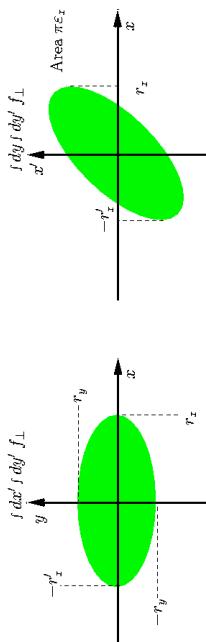
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## Further comments on the KV equilibrium: 2D Projections

### All 2D projections of the KV distribution are uniformly filled ellipses

- ◆ Not very different from what is often observed in experimental measurements and self-consistent simulations of stable beams with strong space-charge
- ◆ Fall-off of distribution at "edges" can be rapid, but smooth, for strong space-charge



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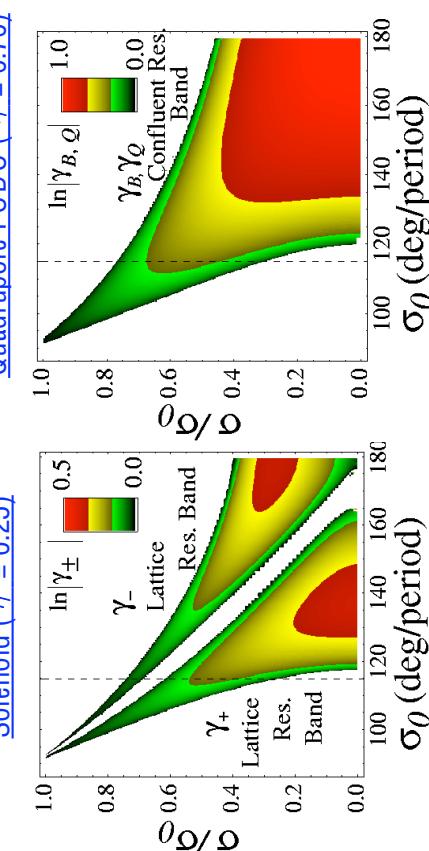
Transverse Equilibrium Distributions 47

## Preview: lecture on Centroid and Envelope Descriptions of Beams.

Instability bands of the KV envelope equation are well understood in periodic focusing channels and must be avoided in machine operation

### Envelope Mode Instability Growth Rates

Solenoid ( $\eta = 0.25$ )



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## Further comments on the KV equilibrium:

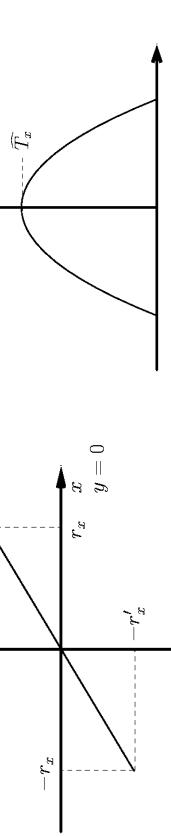
### Angular Spreads: Coherent and Incoherent

Angular spreads within the beam:

#### Coherent (flow):

$$\langle x' \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp x'_\perp f_\perp}{\int d^2x'_\perp f_\perp} = r' \frac{x}{r_x}$$

$$\langle (x' - r'_x x / r_x)^2 \rangle_{\mathbf{x}'_\perp} = \frac{\langle (x' - r'_x x / r_x)^2 \rangle_{\mathbf{x}'_\perp}}{\langle x' \rangle_{\mathbf{x}'_\perp}^2} = \frac{\varepsilon_x^2}{2r_x^2} \left( 1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} \right)$$



◆ Coherent flow required for periodic focusing to conserve charge

◆ Temperature must be zero at the beam edge since the distribution edge is sharp

◆ Parabolic temperature profile is consistent with linear grad P pressure forces in a fluid model interpretation of the (kinetic) KV distribution

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### Further comments on the KV equilibrium:

The KV distribution is the *only* exact equilibrium distribution formed from Courant-Snyder invariants of linear forces valid for periodic focusing channels:

- ♦ Low order properties of the distribution are physically appealing
- ♦ Illustrates relevant Courant-Snyder invariants in simple form
  - later arguments demonstrate that these invariants should be a reasonable approximation for beams with strong space charge
- ♦ KV distribution does not have a 3D generalization [see F. Sacherer, Ph.d. thesis, 1968]

**Strong Vlasov instabilities associated with the KV model render the distribution inappropriate for use in evaluating machines at high levels of detail:**

- ♦ Instabilities are not all physical and render interpretation of results difficult
    - Difficult to separate physical from nonphysical effects in simulations
- Possible Research Problem (unsolved in 40+ years!):**
- Can a valid Vlasov equilibrium be constructed for a *smooth* (non-singular), nonuniform density distribution in a linear, periodic focusing channel?

- ♦ Not clear what invariants can be used or if any can exist
  - Nonexistence proof would also be significant
- ♦ Lack of a smooth equilibrium does not imply that real machines cannot work!

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Because of a lack of theory for a smooth, self-consistent distribution that would be more physically appealing than the KV distribution we will examine smooth distributions in the idealized continuous focusing limit (after an analysis of the **continuous limit of the KV theory**):

- ♦ Allows more classic "plasma physics" like analysis
  - ♦ Illuminates physics of intense space charge
  - ♦ Lack of continuous focusing in the laboratory will prevent over generalization of results obtained
- A 1D analog to the KV distribution called the "Neuffer Distribution" is useful in longitudinal physics**
- ♦ Based on linear forces with a "g-factor" model
  - ♦ Distribution is not singular in 1D
  - ♦ See: J.J. Barnard, lectures on **Longitudinal Physics**

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### Appendix A: Self-Fields of a Uniform Density Elliptical Beam in Free-Space

See handwritten notes

### Appendix B: Canonical Transformation of the KV Distribution

See handwritten notes

#### S4: Continuous Focusing limit of the KV Equilibrium Distribution

Continuous focusing, axisymmetric beam

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Undepressed betatron wavenumber

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon$$

$$r_x = r_y \equiv r_b$$

KV envelope equation

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

immediately reduces to:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

with solution

$$r_b = \left( \frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}$$

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Similarity, the particle equations of motion within the beam are:

$$\begin{aligned} x'' + \left\{ \kappa_x - \frac{2Q}{[r_x + r_y] r_x} \right\} x &= 0 \\ y'' + \left\{ \kappa_y - \frac{2Q}{[r_x + r_y] r_y} \right\} y &= 0 \end{aligned}$$

reduce to

$$\mathbf{x}_\perp'' + k_\beta^2 \mathbf{x}_\perp = 0$$

with solution

$$\mathbf{x}_\perp(s) = \mathbf{x}_\perp i \cos[k_\beta(s - s_i)] + \frac{\mathbf{x}'_\perp i}{k_\beta} \sin[k_\beta(s - s_i)]$$

Space-charge tune depression (rate of phase advance same everywhere,  $L_p$  arb.)

$$\frac{k_\beta}{k_{\beta 0}} = \frac{\sigma}{\sigma_0} = \left( 1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right)^{1/2} \quad 0 \leq \frac{\sigma}{\sigma_0} \leq 1$$

$$\begin{aligned} \varepsilon &\rightarrow 0 & Q &\rightarrow 0 \\ \Rightarrow k_{\beta 0}^2 r_b^2 &= Q & \end{aligned}$$

envelope equation

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Transverse Equilibrium Distributions 54

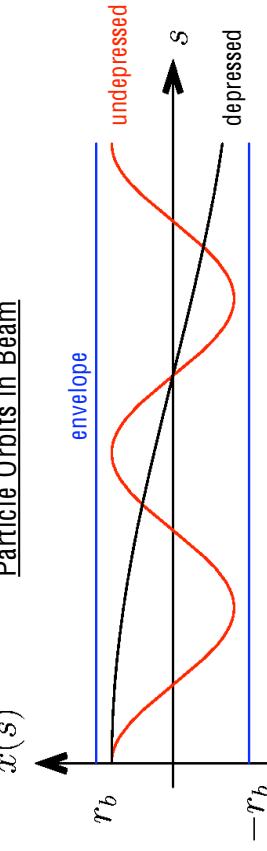
#### Continuous Focusing KV Beam ± Equilibrium Distribution Form

Undepressed and depressed particle orbits in the  $-x$ -plane

$$k_\beta = \frac{\sigma}{\sigma_0} k_{\beta 0} \quad \frac{\sigma}{\sigma_0} = 0.2 \quad = 0 =$$

$x(s)$

Particle Orbits in Beam



Much simpler in details than the periodic focusing case, but qualitatively similar in that space-charge depresses the rate of particle phase advance

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#### Continuous Focusing KV Beam ± Equilibrium Distribution Form

Using  $\lambda = q\pi\hat{n}r_b^2$   $\hat{n} = \text{const}$  density within the beam

$$\delta(\text{const} \cdot x) = \frac{\text{const}}{\delta(x)}$$

the full elliptic beam KV distribution can be expressed as :

- See next slide for steps involved in the form reduction

$$\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta \left[ \left( \frac{x}{r_x} \right)^2 + \left( \frac{r_x x' - r' x}{\varepsilon_x} \right)^2 + \left( \frac{y}{r_y} \right)^2 + \left( \frac{r_y y' - r' y}{\varepsilon_y} \right)^2 - 1 \right] \\ &= \frac{\hat{n}}{2\pi} \delta(H_\perp - H_{\perp b}) \end{aligned}$$

$$\text{where } H_\perp = \frac{1}{2} \mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4} \mathbf{x}_\perp^2$$

$$= \frac{1}{2} \mathbf{x}_\perp'^2 + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$$

$H_{\perp b} \equiv \frac{\varepsilon^2}{2r_b^2} = \text{const}$

(on-axis value 0 ref)

-- Hamiltonian at beam edge

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### // Aside: Steps of derivation

Using:  $\varepsilon_x = \varepsilon_y \equiv \varepsilon$        $\lambda = q\pi\hat{n}r_b^2 = \text{const}$

$$r_x = r_y \equiv r_b = \text{const}$$

$$f_\perp = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 - 1$$

$$= \frac{\hat{n}r_b^2}{\pi\varepsilon^2}\delta\left(\frac{x^2}{r_b^2} + \frac{y^2}{r_b^2} + \frac{r_b^2 x'^2}{\varepsilon^2} + \frac{r_b^2 y'^2}{\varepsilon^2} - 1\right)$$

Using:

$$\delta(\text{const} \cdot x) = \frac{\delta(x)}{\text{const}}$$

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}_\perp^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 - \frac{\varepsilon^2}{2r_b^2}\right)$$

The solution for the potential for the uniform density beam inside the beam is:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} = -\frac{\lambda}{\pi\varepsilon_0 r_b^2} \quad \rightarrow \quad \phi = -\frac{\lambda}{4\pi\varepsilon_0 r_b^2}\mathbf{x}_\perp^2 + \text{const}$$

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The Hamiltonian becomes:

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 - \frac{q\lambda}{4\pi m\gamma_b^3\beta_b^2c^2}\mathbf{x}_\perp^2 + \text{const}$$

$$= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 - \frac{Q}{2r_b^2}\mathbf{x}_\perp^2 + \text{const}$$

$$= \text{const}$$

From the equilibrium envelope equation:

$$k_{\beta 0}^2 = \frac{Q}{r_b^2} + \frac{\varepsilon^2}{r_b^4}$$

The Hamiltonian reduces to:

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 + \text{const}$$

with edge value (turning point with zero angle):

$$H_{\perp b} \equiv \frac{\varepsilon^2}{2r_b^2} + \text{const}$$

Giving (constants are same in Hamiltonian and edge value and subtract out):

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 - \frac{\varepsilon^2}{2r_b^2}\right) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$

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### Equilibrium distribution

$$f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})$$

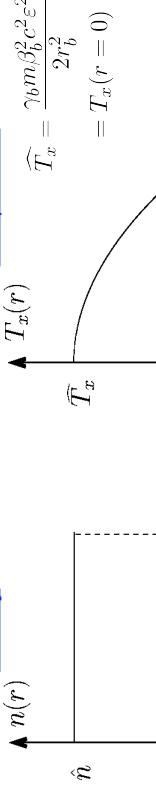
$$\hat{n} = \text{const}$$

then it is straightforward to explicitly calculate (see homework problems)

$$\text{Density: } n = \int d^2x'_\perp f_\perp = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b \leq r \end{cases}$$

$$\text{Temperature: } T_x = \gamma_b m \beta_b^2 c^2 \frac{\int d^2x'_\perp x'^2 f_\perp}{\int d^2x'_\perp f_\perp} = \begin{cases} \widehat{T}_x(1 - r^2/r_b^2), & 0 \leq r < r_b \\ 0, & r_b \leq r \end{cases}$$

### Density



### Temperature



### Continuous Focusing KV Beam ± Comments

For continuous focusing,  $H_\perp$  is a single particle constant of the motion (see problem sets), so it is not surprising that the KV equilibrium form reduces to a delta function form of  $f_\perp(H_\perp)$

♦ Because of the delta-function distribution form, all particles in the continuous focusing KV beam have the same transverse energy with  $H_\perp = H_{\perp b} = \text{const}$

Several textbook treatments of the KV distribution derive continuous focusing versions and then just write down (if at all) the periodic focusing version based on Courant-Snyder invariants. This can create a false impression that the KV distribution is a Hamiltonian-type invariant in the general form.

♦ For non-continuous focusing channels there is no simple relation between Courant-Snyder type invariants and  $H_\perp$

## S5: Equilibrium Distributions in Continuous Focusing Channels

Take

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

- ♦ Real transport channels have  $s$ -varying focusing functions

♦ For a rough correspondence to physical lattices take:  $k_{\beta 0} = \sigma_0 / L_p$

A valid family of **equilibria** can be constructed for any choice of function:

$$f_{\perp} = f_{\perp}(H_{\perp}) \geq 0 \quad H_{\perp} = \frac{1}{2}\mathbf{x}_{\perp}^{\prime 2} + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_{\perp}^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

must be calculated consistently from the (generally nonlinear) **Poisson equation**:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{q}{\epsilon_0} \int d^2x'_{\perp} f_{\perp}(H_{\perp})$$

♦ Solutions generated will be steady-state ( $\partial/\partial s = 0$ )

♦ When  $f_{\perp} = f_{\perp}(H_{\perp})$ , the Poisson equation *only* has axisymmetric solutions with  $\partial/\partial\theta = 0$  [see: Lund, PRSTAB 10, 064203 (2007)]

The Hamiltonian is only equivalent to the Courant-Snyder invariant in continuous focusing (see: **Transverse Particle Equations**). In periodic focusing channels  $\kappa_x(s)$  and  $\kappa_y(s)$  vary in  $s$  and the Hamiltonian is *not* a constant of the motion.

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The ax symmetric Poisson equation simplifies to:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = -\frac{qn}{\epsilon_0} = -\frac{q}{\epsilon_0} \int d^2x'_{\perp} f_{\perp}(H_{\perp})$$

Introduce a **streamfunction**

$$\psi(r) = \frac{1}{2}k_{\beta 0}^2r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \quad r = \sqrt{x^2 + y^2}$$

then

$$H_{\perp} = \frac{1}{2}\mathbf{x}_{\perp}^{\prime 2} + \psi$$

and system axisymmetry can be exploited to calculate the **beam density** (see earlier aside slides on integral symmetries for steps) as:

$$n(r) = \int d^2x'_{\perp} f_{\perp}(H_{\perp}) = 2\pi \int_{\psi}^{\infty} dH_{\perp} f_{\perp}(H_{\perp})$$

The **Poisson equation** can then be expressed in terms of the **stream function** as:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2} \int_{\psi(r)}^{\infty} dH_{\perp} f_{\perp}(H_{\perp})$$

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## Moment properties of continuous focusing equilibrium distributions

Equilibria with *any* valid equilibrium  $f_{\perp}(H_{\perp})$  satisfy the **rms equivalent beam matched beam envelope equation**:

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

- ♦ Describes average radial force balance of particles
- ♦ Uses the result (see J.J. Barnard, **Intro. Lectures**):  $\langle x \partial \phi / \partial x \rangle_{\perp} = -\lambda / (8\pi\epsilon_0)$

where

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \quad \lambda = q \int d^2x_{\perp} \int d^2x'_{\perp} f_{\perp}(H_{\perp})$$

$$r_b^2 = 2\langle r^2 \rangle_{\perp} = \frac{\int_0^{\infty} dr r^3 \int_{\psi}^{\infty} dH_{\perp} f_{\perp}(H_{\perp})}{\int_0^{\infty} dr r \int_{\psi}^{\infty} dH_{\perp} f_{\perp}(H_{\perp})}$$

$$\varepsilon^2 = 2r_b^2 \langle \mathbf{x}_{\perp}^{\prime 2} \rangle_{\perp} = 2r_b^2 \frac{\int_0^{\infty} dr r \int_{\psi}^{\infty} dH_{\perp} (H_{\perp} - \psi) f_{\perp}(H_{\perp})}{\int_0^{\infty} dr r \int_{\psi}^{\infty} dH_{\perp} f_{\perp}(H_{\perp})}$$

$$\langle \dots \rangle_{\perp} = \frac{\int d^2x_{\perp} \int d^2x'_{\perp} \dots f_{\perp}(H_{\perp})}{\int d^2x_{\perp} \int d^2x'_{\perp} f_{\perp}(H_{\perp})}$$

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To characterize a choice of equilibrium function  $f_{\perp}(H_{\perp})$ , the (transformed) Poisson equation must be solved

- ♦ Equation is, in general, *highly* nonlinear rendering the procedure difficult
- ♦ Apply rms equivalent beam picture and interpret in terms of moments
- ♦ Calculate equilibria for a few types of very different functions to understand the likely range of characteristics

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## Parameters used to define the equilibrium function

$$f_{\perp}(H_{\perp})$$

should be cast in terms of

$$Q, \varepsilon, r_b$$

for use in accelerator applications. The rms equivalent beam equations can be used to carry out needed parameter eliminations. Such eliminations can be highly nontrivial due to the nonlinear form of the equations.

A kinetic temperature can also be calculated

$$\langle \dots \rangle_{\mathbf{x}'_{\perp}} \equiv \frac{\int d^2x'_{\perp} \dots f_{\perp}}{\int d^2x'_{\perp} f_{\perp}}$$

$$\langle x'^2 \rangle_T(r) = \frac{1}{2} \int d^2x'_{\perp} \mathbf{x}'_{\perp}^2 f_{\perp}(H_{\perp}) = 2\pi \int_{\psi}^{\infty} dH_{\perp} (H_{\perp} - \psi) f_{\perp}(H_{\perp})$$

which is also related to the emittance,

$$\varepsilon^2 = 16 \langle x'^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} = 4r_b^2 \frac{\int d^2x_{\perp} n T}{\int d^2x_{\perp} n}$$

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## Choices of continuous focusing equilibrium distributions:

Common choices for  $f_{\perp}(H_{\perp})$  analyzed in the literature:

1) **KV** (already covered)

$$f_{\perp} \propto \delta(H_{\perp} - H_{\perp b})$$

$$H_{\perp b} = \text{const}$$

2) **Waterbag (to be covered)**

[see M. Reiser, *Charged Particle Beams*, (1994, 2008)]

$$f_{\perp} \propto \Theta(H_{\perp b} - H_{\perp})$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

3) **Thermal (to be covered)**

[see M. Reiser, Davidson, *Noneutral Plasmas*, 1990]

$$f_{\perp} \propto \exp(-H_{\perp}/T)$$

$$T = \text{const} > 0$$

Infinity of choices can be made for an infinity of papers!  
 ♦ Fortunately, range of behavior can be understood with a few reasonable choices

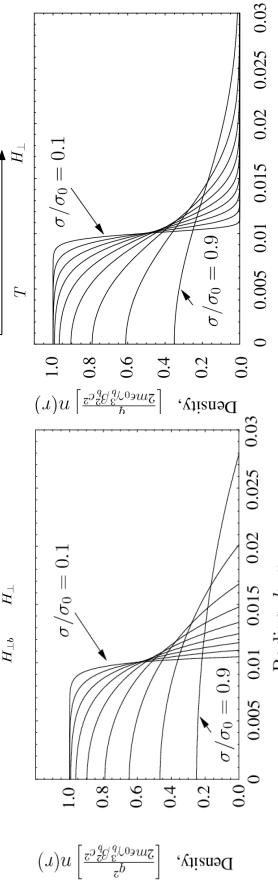
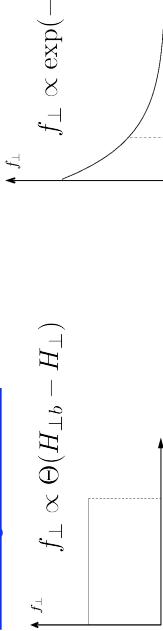
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 Transverse Equilibrium Distributions 66

**Preview of what we will find:** When relative space-charge is strong, all smooth equilibrium distributions expected to look similar

Constant charge and focusing:  $Q = 10^{-4}$   $k_{\beta 0}^2 = \text{const}$

Vary relative space-charge strength:  $\sigma/\sigma_0 = 0.1, 0.2, \dots, 0.9$

Waterbag Distribution



Edge shape varies with distribution choice, but cores similar when  $\sigma/\sigma_0$  small

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**S6: Continuous Focusing: The Waterbag Equilibrium Distribution:**  
 [Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, 2008);  
 and Review: Lund, Kikuchi, and Davidson, PRSTAB, to be published (2008)]

Waterbag distribution:

$$f_{\perp}(H_{\perp}) = f_0 \Theta(H_b - H_{\perp}) \quad f_0 = \text{const}$$

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

The physical edge radius  $r_e$  of the beam will be related to the edge Hamiltonian:

$$H_{\perp}|_{r=r_e} = H_b \quad \text{Note (generally): } \begin{cases} r_e \neq r_b \equiv 2\langle x^2 \rangle^{1/2} \\ r_e > r_b \end{cases}$$

Using previous formulas the equilibrium density can then be calculated as:

$$H_{\perp} = \mathbf{x}_{\perp}^2/2 + \psi \quad \psi = k_{\beta 0}^2 r^2 / 2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$$

$$n(r) = \int d^2x'_{\perp} f_{\perp} = 2\pi f_0 \begin{cases} H_b - \psi(r), & \psi < H_b, \\ 0, & \psi > H_b. \end{cases}$$

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The Poisson equation of the equilibrium can be expressed within the beam ( $r < r_e$ ) as:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) - k_0^2 \psi = 2k_{\beta 0}^2 - k_0^2 H_b$$

$$k_0^2 \equiv \frac{2\pi q^2 f_0}{\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const}$$

This is a modified Bessel function equation and the solution within the beam regular at the origin  $r = 0$  and satisfying  $\psi(r = r_e) = H_b$  is given by

$$\psi(r) = H_b - 2 \frac{k_{\beta 0}^2}{k_0^2} \left[ 1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)} \right]$$

where  $I_\ell(x)$  is a modified Bessel function of order  $\ell$

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The density is then expressible within the beam ( $r < r_e$ ) as:

$$n(r) = 4\pi f_0 \frac{k_{\beta 0}^2}{k_0^2} \left[ 1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)} \right]$$

$$= \frac{2\epsilon_0 m \gamma_b^2 \beta_b^2 c^2 k_{\beta 0}^2}{q^2} \left[ 1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)} \right]$$

Similarly, the local beam temperature within the beam can be calculated as:

$$T_x(r) = \langle x'^2 \rangle_{\mathbf{x}'_\perp} = \frac{k_{\beta 0}^2}{k_0^2} \left[ 1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)} \right]$$

$$\propto n(r)$$

The proportionality between the temperature ( $T_x$ ) and the density ( $n$ ) is a consequence of the waterbag equilibrium distribution choice and is *not* a general feature of continuous focusing.

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The waterbag distribution expression can now be expressed as:

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp) = f_0 \Theta \left( 2 \frac{k_{\beta 0}^2}{k_0^2} \left[ 1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)} \right] - \frac{1}{2} \mathbf{x}'_\perp^2 \right)$$

♦ The edge Hamiltonian value  $H_b$  has been eliminated

♦ Parameters are:

$f_0$  .... distribution normalization

$k_{\beta 0}/k_0$  .... scaled edge radius

$\frac{2\ell}{x} I_\ell(x) = I_{\ell+1}(x) - I_{\ell-1}(x)$ ,

$r_b^2 = 2 \langle r'^2 \rangle_\perp = 2 \int_0^{r_e} dr' r'^2 n(r)$

Parameters preferred for accelerator applications:

$$k_{\beta 0}, Q, \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints to eliminate parameters in terms of our preferred set will now be derived.

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Parameters constraints for the waterbag equilibrium beam

First calculate the beam line-charge:

$$\lambda = 2\pi q \int_0^{r_e} dr' r n(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2 \left[ 1 - \frac{2}{k_0 r_e} \frac{I_1(k_0 r_e)}{I_0(k_0 r_e)} \right]$$

$$\lambda = 2\pi q \int_0^{r_e} dr' r n(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}$$

here we have employed the modified Bessel function identities ( $\ell$  integer):

$$\frac{d}{dx} [x^\ell I_\ell(x)] = x^\ell I_{\ell-1}(x),$$

$$-\frac{2\ell}{x} I_\ell(x) = I_{\ell+1}(x) - I_{\ell-1}(x),$$

Similarly, the beam rms edge radius can be explicitly calculated as:

$$r_b^2 = 2 \langle r'^2 \rangle_\perp = 2 \int_0^{r_e} dr' r^3 n(r)$$

$$\left( \frac{r_b}{r_e} \right)^2 = \frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2} \left[ 2 + (k_0 r_e) \frac{I_3(k_0 r_e)}{I_2(k_0 r_e)} \right]$$

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The **pervenace** is then calculated as:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = (k_{\beta 0} r_e)^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}$$

The edge and pervenace equations can then be combined to obtain a parameter constraint relating  $k_0 r_e$  to desired system parameters:

$$\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2} \left[ 2 \frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} \frac{I_3(k_0 r_e)}{I_2^2(k_0 r_e)} \right]$$

Here, any of the 3 system parameters on the LHS may be eliminated using the matched beam envelope equation to effect alternative parameterizations:

$$k_{\beta 0}^2 r_b^2 - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0 \quad \rightarrow \quad \text{eliminate any of: } k_{\beta 0}^2, r_b, Q$$

The rms equivalent beam concept can also be applied to show that:

$$\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{1}{1 - (\sigma/\sigma_0)^2}$$

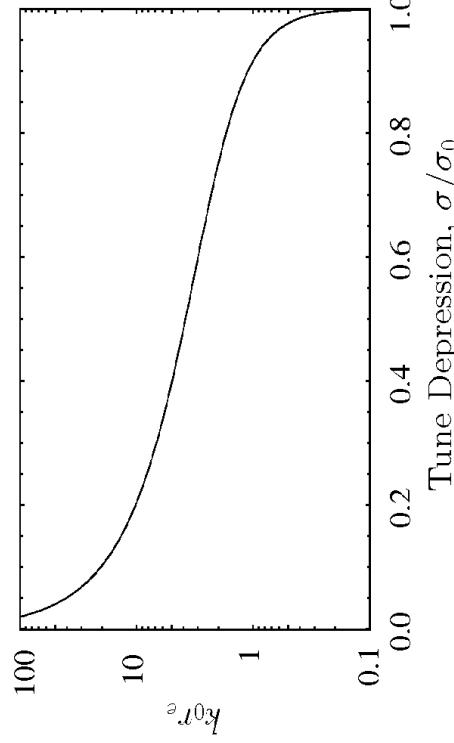
- **rms equivalent KV measure of  $\sigma/\sigma_0$**
- Space-charge really nonlinear and the Waterbag equilibrium has a spectrum of  $\sigma$

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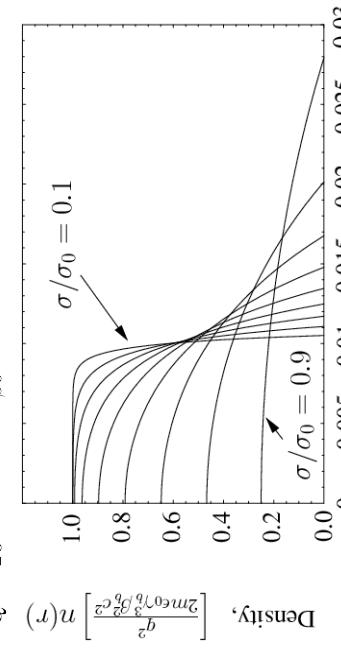
The constraint is plotted over the full range of effective space-charge strength:

$$\frac{1}{1 - (\sigma/\sigma_0)^2} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2} \left[ 2 \frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} + (k_0 r_e) \frac{I_0(k_0 r_e)}{I_2^2(k_0 r_e)} \right]$$



**Use parameter constraints to plot properties of waterbag equilibrium**

- 1) Density and temperature profile at fixed line charge and focusing strength  
 $Q = 10^{-4}$



For a KV equilibrium,  $s_b$  and  $\sigma/\sigma_0$  are simply related:

$$s_b = 1 - \left( \frac{\sigma}{\sigma_0} \right)^2$$

For a waterbag equilibrium,  $s_b$  and  $k_0 r_e$  (from which  $\sigma/\sigma_0$  can be calculated) are related by:

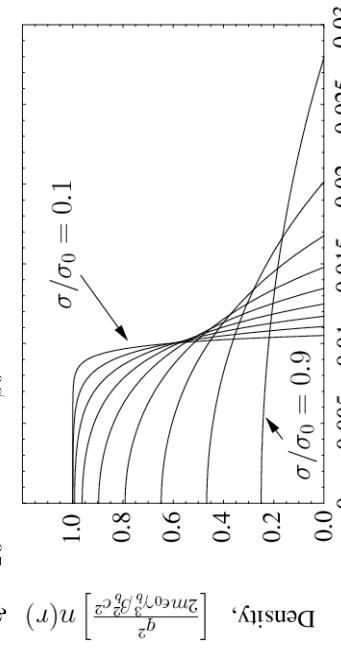
$$s_b = 1 - \frac{1}{I_0(k_0 r_e)}$$

Generally, for smooth (non-KV) equilibria,  $s_b$  turns out to be a logarithmically insensitive parameter for strong space-charge strength (see tables in S6 and S7) //

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**Use parameter constraints to plot properties of waterbag equilibrium**

- 1) Density and temperature profile at fixed line charge and focusing strength  
 $Q = 10^{-4}$



For a KV equilibrium,  $s_b$  and  $\sigma/\sigma_0$  are simply related:

$$s_b = 1 - \left( \frac{\sigma}{\sigma_0} \right)^2$$

For a waterbag equilibrium,  $s_b$  and  $k_0 r_e$  (from which  $\sigma/\sigma_0$  can be calculated) are related by:

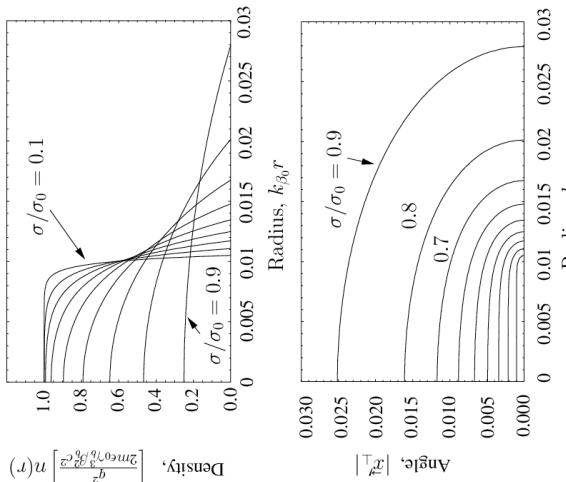
- Parabolic density for weak space-charge and flat in the core out to a sharp edge for strong space charge
- For the waterbag equilibrium, temperature  $(\sigma/\sigma_0)$  is proportional to density  $(s_b)$
- so the same curves apply for  $(s_b)$

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Transverse Equilibrium Distributions 76

Transverse Equilibrium Distributions 76

2) Phase-space boundary of distribution at fixed line charge and focusing strength  
 $Q = 10^{-4}$   
 $k_{\beta 0}^2 = \text{const}$



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3) Summary of scaled parameters for example plots:

$\sigma/\sigma_0$	$s_b$	$\frac{k_{\beta 0}^2 c^2}{Q}$	$k_0 r_e$	$\frac{r_e}{r_b}$	$\frac{k_0}{k_{\beta 0}}$	$10^3 \times k_{\beta 0} \varepsilon_b$	$Q = 10^{-4}$
0.9	0.2502	5.2633	1.112	1.217	39.81	0.4737	
0.8	0.4666	2.778	1.709	1.208	84.87	0.2222	
0.7	0.6477	1.961	2.304	1.197	137.5	0.1373	
0.6	0.7916	1.563	2.979	1.183	201.5	0.09375	
0.5	0.8968	1.333	3.821	1.166	283.8	0.06667	
0.4	0.9626	1.190	4.978	1.144	398.7	0.04762	
0.3	0.9928	1.099	6.789	1.118	579.3	0.03297	
0.2	0.9997	1.042	10.25	1.085	925.6	0.02083	
0.1	1.0000	1.010	20.38	1.046	1938.	0.01010	

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### S7: Continuous Focusing: The Thermal Equilibrium Distribution:

[Davidson, Physics of Nonneutral Plasma, Addison Wesley (1990) and Reiser, Theory and Design of Charged Particle Beams, Wiley (1994, 2008)]

In an infinitely long continuous focusing channel, collisions will eventually relax the beam to **thermal equilibrium**. The Fokker-Planck equation predicts that the unique Maxwell-Boltzmann distribution describing this limit is:

$$\lim_{s \rightarrow \infty} f_{\perp} \propto \exp\left(-\frac{H_{\text{rest}}}{T}\right)$$

$H_{\text{rest}} = \frac{1}{2} \mathbf{x}_{\perp}^{\prime 2} + \frac{1}{2} k_{\beta 0}^2 \mathbf{x}_{\perp}^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$   
 $= \frac{1}{2} \mathbf{x}_{\perp}^{\prime 2} + \psi$

The density can then be conveniently calculated in terms of a scaled stream function:

$$n(r) = \int d^2 x'_{\perp} f_{\perp} = \hat{n} e^{-\tilde{\psi}}$$

$$\tilde{\psi}(r) \equiv \frac{m\gamma_b \beta_b^2 c^2 \psi}{T} = \frac{1}{T} \left( \frac{m\gamma_b \beta_b^2 c^2 k_{\beta 0}^2}{2} r^2 + \frac{q\phi}{\gamma_b^2} \right)$$

and the - and -temperatures are equal and spatially uniform with:

$$T_x = \gamma_b m \beta_b^2 c^2 \frac{\int d^2 x'_{\perp} x'^2 f_{\perp}}{\int d^2 x'_{\perp} f_{\perp}} = T = \text{const}$$

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## Scaled Poisson equation for continuous focusing thermal equilibrium

To describe the thermal equilibrium density profile, the **Poisson equation** must be solved. In terms of the scaled streamfunction:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \tilde{\psi}}{\partial \rho} \right) = 1 + \Delta - e^{-\tilde{\psi}}$$

$$\tilde{\psi}(\rho = 0) = 0 \quad \frac{\partial \tilde{\psi}}{\partial \rho}(\rho = 0) = 0$$

Here,

$$\lambda_D = \left( \frac{\epsilon_0 T}{q^2 \hat{n}} \right)^{1/2}$$

Debye length formed from the peak, on-axis beam density

$$\omega_p \equiv \left( \frac{q^2 \hat{n}}{\epsilon_0 m} \right)^{1/2}$$

Plasma frequency formed from on-axis beam density

$$\Delta = \frac{2 \gamma_b^3 \beta_b^2 c^2 k_{\beta 0}^2}{\omega_p^2} - 1$$

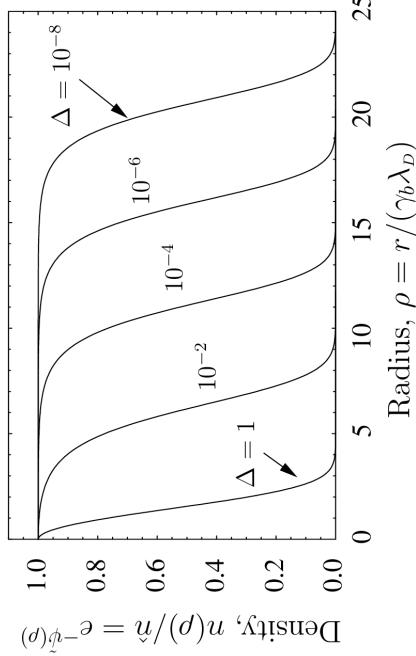
Dimensionless parameter relating the ratio of applied to space-charge defocusing forces

- ♦ Equation is highly nonlinear, but can be solved (approximately) analytically
- ♦ Scaled solutions depend only on the single dimensionless parameter #

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## Numerical solution of scaled thermal equilibrium Poisson equation in terms of a normalized density



♦ Equation is highly nonlinear and must, in general, be solved numerically

- Dependence on # is very sensitive

- For small #, the beam is nearly uniform in the core

- ♦ Edge fall-off is always in a few Debye lengths when # is small
- ♦ Edge becomes very sharp at fixed beam line-charge

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### /// Aside: Approximate Analytical Solution for the Thermal Equilibrium

Density/Potential

$$N \equiv \frac{n}{\hat{n}} = e^{-\tilde{\psi}}$$

the equilibrium Poisson equation can be equivalently expressed as:

$$\frac{\partial^2 N}{\partial \rho^2} - \frac{1}{N} \left( \frac{\partial N}{\partial \rho} \right)^2 + \frac{1}{\rho} \frac{\partial N}{\partial \rho} = N^2 - (1 + \Delta) N$$

$$N(\rho = 0) = 1$$

$$\left. \frac{\partial N}{\partial \rho} \right|_{\rho=0} = 0$$

This equation has been analyzed to construct limiting form analytical solutions for both large and small  $\Delta$  [see: Starkev and Lund, PoP 15, 043101 (2008)].

- ♦ **Large**  $\Delta$  solution => warm beam => Gaussian-like radial profile
- ♦ **Small**  $\Delta$  solution => cold beam => Flat core, bell shaped profile
- Highly nonlinear structure, but approx solution has very high accuracy out to where the density becomes exponentially small!

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Large  $\Delta$  solution:

$$N \simeq \exp \left[ -\frac{1 + \Delta}{4} \rho^2 \right]$$

- ♦ Accurate for  $\Delta \gtrsim 0.1$  [For full error spec. see: PoP 15, 043101 (2008)]

Small  $\Delta$  solution:

$$N \simeq \frac{(1 + \frac{1}{2} \Delta + \frac{1}{24} \Delta^2)^2}{\{1 + \frac{1}{2} \Delta I_0(\rho) + \frac{1}{24} [\Delta I_0(\rho)]^2\}^2}$$

- ♦ Highly accurate for  $\Delta \lesssim 0.1$  [For full error spec. see: PoP 15, 043101 (2008)]

Special numerical methods have also been developed to calculate  $\tilde{\psi} = -\ln N$  to arbitrary accuracy for any value of  $\Delta$ , however small! [see: Lund, Kikuchi, and Davidson, PRSTAB, to be published, (2008) Appendices F, G]

- ♦ Extreme flatness of solution for small  $\Delta \lesssim 10^{-8}$  creates numerical precision problems that require special numerical methods to address
- ♦ Method was used to verify accuracy of small  $\Delta$  solution above

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## Parameters constraints for the thermal equilibrium beam

Parameters employed in  $f_{\perp}(H_{\perp})$  to specify the equilibrium are (+ kinematic factors):  $\hat{n}, T, \Delta$

Parameters preferred for accelerator applications:

$$k_{\beta 0}, Q, \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints can be calculated directly from the equilibrium.

$$Q = \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) \int_0^{\infty} d\rho \rho e^{-\tilde{\psi}}$$

Integral function  
of  $\Delta$  only

$$k_{\beta 0}^2 \varepsilon_b = 4 \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) \left[ 4 \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) + Q \right]$$

$$k_{\beta 0}^2 = \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) \frac{1 + \Delta}{2(\gamma_b \lambda_D)^2}$$

Also useful,

$$\varepsilon_b^2 = 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_{\perp} = 4 \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) r_b^2$$

$$r_b^2 = 4 \langle x^2 \rangle_{\perp} = \frac{1}{k_{\beta 0}^2} \left[ 4 \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) + Q \right]$$

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Example of derivation steps applied to derive previous constraint equations:

$$\text{Line charge: } \lambda = \frac{\gamma_b^2 T}{2q} \int_0^{\infty} d\rho \rho e^{-\tilde{\psi}}$$

$$\text{rms edge radius: } r_b^2 = 4 \langle x^2 \rangle_{\perp} = 2 \gamma_b^2 \lambda_D^2 \frac{\int_0^{\infty} d\rho \rho^3 e^{-\tilde{\psi}}}{\int_0^{\infty} d\rho \rho e^{-\tilde{\psi}}}$$

rms edge emittance:

$$\begin{aligned} \varepsilon_b^2 &= \varepsilon_x^3 = 16 [\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle x x' \rangle_{\perp}^2] \\ &= 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_{\perp} = 4 \left( \frac{T}{\gamma_b m \beta_b^2 c^2} \right) r_b^2 \end{aligned}$$

Matched envelope equation:

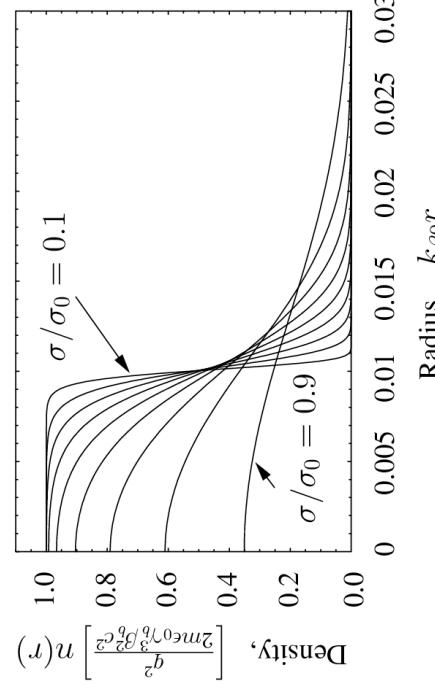
$$r_b' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0$$

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2) Density profile at fixed line charge and focusing strength

$$Q = 10^{-4} \quad k_{\beta 0}^2 = \text{const}$$



- ♦ Density profile changes with scaled T
  - Low values yields a flat-top =>  $\sigma/\sigma_0 \rightarrow 0$
  - High values yield a Gaussian like profile =>  $\sigma/\sigma_0 \rightarrow 1$

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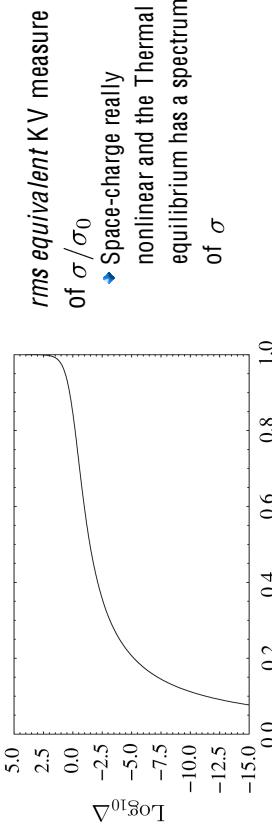
These constraints must, in general, be solved numerically

♦ Useful to probe system sensitivities in relevant parameters

Examples:

1) rms equivalent beam tune depression as a function of #

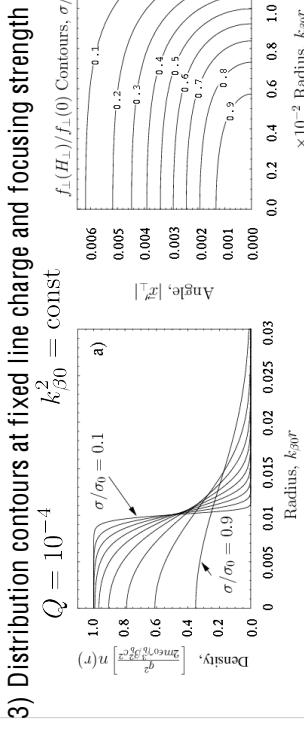
$$\frac{\sigma}{\sigma_0} = \sqrt{1 - \frac{Q}{k_{\beta 0}^2 r_b^2}} = \left\{ 1 - \frac{[\int_0^{\infty} d\rho \rho e^{-\tilde{\psi}}]^2}{(1 + \Delta) \int_0^{\infty} d\rho \rho^3 e^{-\tilde{\psi}}} \right\}^{1/2} \quad \text{R.H.S function of \# only}$$



- ♦ Small rms equivalent tune depression corresponds to extremely small values of #
  - Special numerical methods generally must be employed to calculate equilibrium

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- ♦ Particles will move approximately force-free till approaching the edge where it is rapidly bent back (see Debye screening analysis this lecture)

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### Scaled parameters for examples 2) and 3)

$\sigma/\sigma_0$	$\Delta$	$s_b$	$k_{\beta 0} \gamma_b \lambda_D$	$\frac{T}{m \gamma_b \beta_b^2 c^2}$	$10^3 \times k_{\beta 0} \varepsilon_b$	$Q = 10^{-4}$
0.9	1.851	0.3508	12.33	$1.065 \times 10^{-4}$	0.4737	
0.8	$6.382 \times 10^{-1}$	0.6104	6.034	$4.444 \times 10^{-5}$	0.2222	
0.7	$2.649 \times 10^{-1}$	0.7906	3.898	$2.402 \times 10^{-5}$	0.1373	
0.6	$1.059 \times 10^{-1}$	0.9043	2.788	$1.406 \times 10^{-5}$	0.09375	
0.5	$3.501 \times 10^{-2}$	0.9662	2.077	$8.333 \times 10^{-6}$	0.06667	
0.4	$7.684 \times 10^{-3}$	0.9924	1.549	$4.762 \times 10^{-6}$	0.04762	
0.3	$6.950 \times 10^{-4}$	0.9993	1.112	$2.473 \times 10^{-6}$	0.03297	
0.2	$6.389 \times 10^{-6}$	1.0000	0.7217	$1.042 \times 10^{-6}$	0.02083	
0.1	$4.975 \times 10^{-12}$	1.0000	0.3553	$2.525 \times 10^{-7}$	0.01010	

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### Comments on continuous focusing thermal equilibria

From these results it is not surprising that the KV model works well for real beams with strong space-charge (i.e. rms equivalent  $\sigma/\sigma_0$  small) since the edges of a smooth thermal distribution become sharp

- ♦ Thermal equilibrium likely overestimates the edge with since  $T = \text{const}$ , whereas a real distribution likely becomes colder near the edge

However, the beam edge contains strong nonlinear terms that will cause deviations from the KV model

- ♦ Nonlinear terms can radically change the stability properties (stabilize fictitious higher order KV modes)
- ♦ Smooth distributions contain a spectrum of particle oscillation frequencies that are amplitude dependent

### S8: Continuous Focusing: Debye Screening in a Thermal Equilibrium Beam

[Davidson, *Physics of Nonneutral Plasmas*, Addison Wesley (1990)]

We will show that space-charge and the applied focusing forces of the lattice conspire together to **Debye screen interactions** in the core of a beam with high space-charge intensity

- ♦ Will systematically derive the Debye length employed by J.J. Barnard in the [Introductory Lectures](#)
- ♦ The applied focusing forces are analogous to a stationary neutralizing species in a plasma

// Review:

Free-space field of a "bare" test line-charge  $\lambda_t$  at the origin  $r = 0$

$$\rho(r) = \lambda_t \frac{\delta(r)}{2\pi r} \quad E_r = -\frac{\partial \phi}{\partial r} = -\frac{\lambda_t}{2\pi \epsilon_0} \frac{\delta(r)}{r}$$

solution (use Gauss' theorem) shows long-range interaction

$$\phi = -\frac{\lambda_t}{2\pi \epsilon_0} \ln(r) + \text{const}$$

Place a small test line charge at  $r = 0$  in a thermal equilibrium beam:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = -\frac{q}{\epsilon_0} \int d^2 x'_\perp f_\perp(H_\perp) - \frac{\lambda_t}{2\pi\epsilon_0} \frac{\delta(r)}{r}$$

### Thermal Equilibrium

### Test Line-Charge

$$\begin{aligned} \phi &= \phi_0 + \delta\phi & \phi_0 &= \text{Thermal Equilibrium potential with no test line-charge} \\ \delta\phi &= \text{Perturbed potential from test line-charge} & \text{Assume thermal equilibrium adapts adiabatically to the test line-charge:} \end{aligned}$$

$$n(r) = \int d^2 x'_\perp f_\perp(H_\perp) = \hat{n} e^{-\tilde{\psi}} \simeq \hat{n} e^{-\tilde{\psi}_0(r)} e^{-q\delta\phi/(\gamma_b T)}$$

$$\left| \frac{q\delta\phi}{\gamma_b^2 T} \right| \ll 1$$

$$\simeq \hat{n} e^{-\tilde{\psi}_0(r)} \left( 1 - \frac{q\delta\phi}{\gamma_b^2 T} \right)$$

$$\text{Yields: } \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta\phi}{\partial r} \right) = -\frac{q^2}{\epsilon_0 \gamma_b^2 T} \hat{n} e^{-\tilde{\psi}_0(r)} - \frac{\lambda_t}{2\pi\epsilon_0} \frac{\delta(r)}{r}$$

Assume a relatively cold beam so the density is flat near the test line-charge:

$$\hat{n} e^{-\tilde{\psi}_0(r)} \simeq \hat{n}$$

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This gives:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta\phi}{\partial r} \right) - \frac{\delta\phi}{\gamma_b^2 \lambda_D^2} = -\frac{\lambda_t}{2\pi\epsilon_0} \frac{\delta(r)}{r}$$

$$\text{Set: } \lambda_D = \left( \frac{\epsilon_0 T}{q^2 \hat{n}} \right)^{1/2} = \begin{array}{l} \text{Debye radius formed from peak,} \\ \text{on-axis beam density} \end{array}$$

Derive a general solution by connecting solution very near the test charge with the general solution for  $r$  nonzero:

Near solution: ( $r \rightarrow 0$ )

$$\frac{\delta\phi}{\gamma_b^2 \lambda_D^2} \quad \text{Negligible} \quad \Rightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \delta\phi}{\partial r} \right) = -\frac{\lambda_t}{2\pi\epsilon_0} \frac{\delta(r)}{r}$$

The free-space solution can be immediately applied:

$$\delta\phi \simeq -\frac{\lambda_t}{2\pi\epsilon_0} \ln(r) + \text{const}$$

$$r \rightarrow 0$$

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Comparison shows that we must choose for connection to the near solution and regularity at infinity:

$$\boxed{C_1 = 0}$$

$$C_2 = \frac{\lambda_t}{2\pi\epsilon_0}$$

General solution shows Debye screening of test charge in the core of the beam:

$$\delta\phi = \frac{\lambda_t}{2\pi\epsilon_0} K_0 \left( \frac{r}{\gamma_b \lambda_D} \right)$$

$$\simeq \frac{\lambda_t}{2\sqrt{2\pi\epsilon_0}} \frac{1}{\sqrt{r/(\gamma_b \lambda_D)}} e^{-r/(\gamma_b \lambda_D)} \quad r \gg \gamma_b \lambda_D$$

### Connection and General Solution:

Use limiting forms:

- $\rho \ll 1$
- $I_0(\rho) \rightarrow 1 + \Theta(\rho^2)$
- $K_0(\rho) \rightarrow -[\ln(\rho/2) + 0.5772 \dots + \Theta(\rho^2)]$
- $I_0(\rho) \rightarrow \frac{e^\rho}{\sqrt{2\pi\rho}} [1 + \Theta(1/\rho)]$
- $K_0(\rho) \rightarrow \sqrt{\frac{\pi}{2\rho}} [1 + \Theta(1/\rho)]$
- Screened interaction does not require overall charge neutrality!
- Beam particles redistribute to screen bare interaction
- Beam behaves as a plasma and expect similar collective waves etc.
- ♦ Same result for all smooth equilibrium distributions and in 1D, 2D, and 3D
- Reason why lower dimension models can get the "right" answer for collective interactions in spite of the Coulomb force varying with dimension
- ♦ Explains why the radial density profile in the core of space-charge dominated beams are expected to be flat

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## S9: Continuous Focusing: The Density Inversion Theorem

Shows and dependencies are strongly connected in an equilibrium

$$f_{\perp} = f_{\perp}(H_{\perp}) \quad H_{\perp} = \frac{1}{2}\mathbf{x}_{\perp}^{\prime 2} + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_{\perp}^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

calculate the beam density

$$n(r) = \int d^2x'_{\perp} f_{\perp}(H_{\perp}) = 2\pi \int_0^{\infty} dU f_{\perp}(U + \psi(r))$$

differentiate:

$$\frac{\partial n}{\partial \psi} = 2\pi \int_0^{\infty} dU \frac{\partial}{\partial \psi} f_{\perp}(U + \psi) = 2\pi \int_0^{\infty} dU \frac{\partial}{\partial U} f_{\perp}(U + \psi)$$

$$= 2\pi \lim_{U \rightarrow \infty} f_{\perp}(U + \psi) - 2\pi f_{\perp}(\psi)$$


$$f_{\perp}(H_{\perp}) = -\frac{1}{2\pi} \frac{\partial n}{\partial \psi} \Big|_{\psi=H_{\perp}}$$

Assume that  $n(r)$  is specified, then the Poisson equation can be integrated:

$$\psi(r) - \frac{q\phi(r=0)}{m\gamma_b^3\beta_b^2c^2} = \frac{1}{2}k_{\beta 0}^2r^2 - \frac{q}{m\gamma_b^3\beta_b^2c^2\epsilon_0} \int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{r}' \tilde{r}' n(\tilde{r})$$

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For  $n(r) = \text{const}$   $\int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{r}' \tilde{r}' n(\tilde{r}) \propto r^2$   
 This suggests that  $(n)$  is monotonic in  $r$  when  $d(n(r)/dr)$  is monotonic. Apply the chain rule:

### Density Inversion Theorem

$$f_{\perp}(H_{\perp}) = -\frac{1}{2\pi} \frac{\partial n}{\partial \psi} \Big|_{\psi=H_{\perp}} = -\frac{1}{2\pi} \frac{\partial n(r)/\partial r}{\partial \psi(r)/\partial r} \Big|_{\psi=H_{\perp}}$$

$$\psi(r) = \frac{1}{2}k_{\beta 0}^2r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

For specified monotonic  $n(r)$  the density inversion theorem can be applied with the Poisson equation to calculate the corresponding equilibrium  $f_{\perp}(H_{\perp})$

Comments on density inversion theorem:

- ◆ Shows that the and dependence of the distribution are *inextricably linked* for an equilibrium distribution function  $f_{\perp}(H_{\perp})$ 
  - Not so surprising -- equilibria are highly constrained
- ◆ If  $df_{\perp}(H_{\perp})/dH_{\perp} \leq 0$  then the kinetic stability theorem (see: S.M. Lund, lectures on Transverse Kinetic Stability) shows that the equilibrium is also stable

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## // Example: Application of the inversion theorem to the KV equilibrium

$$n = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases} \longrightarrow \frac{\partial n}{\partial r} = -\hat{n}\delta(r - r_b)$$

property of delta-function:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}}$$

$$f(x_i) = 0$$

$x_i$  is root of  $f$

$$\frac{\partial n}{\partial \psi} = \frac{\partial n/\partial r}{\partial \psi/\partial r}$$

$$= -\frac{\hat{n}\delta(r - r_b)}{\partial \psi/\partial r}$$

$$= -\frac{\hat{n}\delta(r - r_b)}{\partial \psi/\partial r|_{r=r_b}}$$

$$= -\hat{n}\delta(\psi(r) - \psi(r_b))$$

use:  $\psi(r_b) = H_{\perp}|_{\mathbf{x}'_{\perp}=0} = H_{\perp b}$

$$f_{\perp}(H_{\perp}) = -\frac{1}{2\pi} \frac{\partial n}{\partial \psi} \Big|_{\psi=H_{\perp}} = \frac{\hat{n}}{2\pi} \delta(H_{\perp} - H_{\perp b})$$

Expected KV form //

Similar application of derivatives with respect to Courant-Snyder invariants can derive the needed form for the KV distribution of an elliptical beam without guessing.

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## S10: Comments on the Plausibility of Smooth, Vlasov Equilibria in Periodic Transport Channels

The KV and continuous models are the only (or related to simple transforms thereof) known exact beam equilibria. Both suffer from idealizations that render them inappropriate for use as initial distribution functions for detailed modeling of stability in real accelerator systems:

- ◆ KV distribution has an unphysical singular structure giving rise to collective instabilities with unphysical manifestations
  - Low order properties (envelope and some features of low-order plasma modes) are physical and very useful in machine design
  - ◆ Continuous focusing is inadequate to model real accelerator lattices with periodic or -varying focusing forces
    - Kicked oscillator intrinsically different than a continuous oscillator

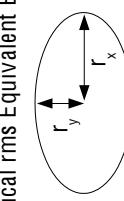
There is much room for improvement in this area, including study if smooth equilibria exist in periodic focusing and implications if no exact equilibria exist.

Transverse Equilibrium Distributions 100

Large envelope flutter associated with strong focusing can result in a rapid high-order oscillating force imbalance acting on edge particles of the beam

#### Temperature Flutter

Elliptical rms Equivalent Beam



Example Systems	$(r_{\max}/r_{\min})^2$
AG Trans: $\beta_0 = 60^\circ$	$\sim 2.5$
AG Trans: $\beta_0 = 100^\circ$	$\sim 4.9$

$$\varepsilon_x^2 \propto T_x r_x^2 \simeq \text{const} \implies T_x \propto \frac{1}{r_x^2} \quad \text{Matching Section} \quad \sim 15 \text{ Possible}$$

#### Characteristic Plasma Frequency of Collective Effects

##### Continuous Focusing Estimate

$$\sigma_{\text{plasma}} \sim \frac{L_p}{r_b} \sqrt{\frac{2Q}{m_p c^2}} \quad \text{Typical: } \sigma_{\text{plasma}} \sim 105^\circ/\text{period}$$

- ◆ Temperature asymmetry in beam will rapidly fluctuate with lattice periodicity
  - Converging plane => Warmer
  - Diverging plane => Colder
- ◆ Collective plasma wave response slower than lattice frequency
  - Beam edge will not be able to adapt rapidly enough
  - Collective waves will be launched from lack of local force balance near the edge

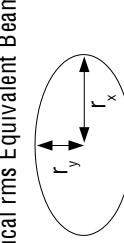
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Transverse Equilibrium Distributions 101

The continuous focusing equilibrium distribution suggests that varying Debye screening together with envelope flutter would require a rapidly adapting beam edge in a smooth, periodic equilibrium beam distribution

#### Temperature Flutter

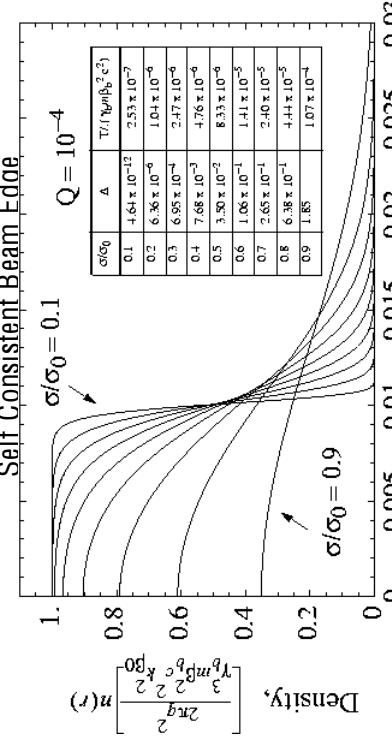
Elliptical rms Equivalent Beam



$$f_\perp = \frac{m \gamma_b \beta_b^2 c^2 \hat{n}}{2\pi T} \exp\left(-\frac{m \gamma_b \beta_b^2 c^2 H_\perp}{T}\right)$$

#### Continuous Focusing Thermal Equilibrium Beam

#### Self Consistent Beam Edge



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Transverse Equilibrium Distributions 102

These slides will be corrected and expanded for reference and any future editions of the US Particle Accelerator School class:

**Beam Physics with Intense Space Charge**, by J.J. Barnard and S.M. Lund

Corrections and suggestions are welcome. Contact:

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It is clear from these considerations that if smooth equilibrium beam distributions exist for periodic focusing, then they are highly nontrivial

Would a **nonexistence** of an equilibrium distribution be a problem:

- ◆ Real beams are born off a source that can be simulated
- ◆ Propagation length can be relatively small in linear
- ◆ Transverse confinement can exist without an equilibrium
  - Particles can turn at large enough radii forming an edge
  - Edge can oscillate from lattice period to lattice period without pumping to large excursions
- ◆ **Might not preclude long propagation with preserved statistical beam quality**

Even approximate equilibria would help sort out complicated processes:

- ◆ Reduce transients and fluctuations can help understand processes in simplest form
  - Allows more plasma physics type analysis and advances
- ◆ Beams in Vlasov simulations are often observed to settle down to a fairly regular state after an initial transient evolution
  - Extreme phase mixing leads to an effective relaxation

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Transverse Equilibrium Distributions 104

## References: For more information see:

- M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, 2008)
- R. Davidson, *Theory of Nonneutral Plasmas*, Addison-Wesley (1989)
- R. Davidson and H. Qin, Physics of Intense Charged Particle Beams in High Energy Accelerators, World Scientific (2001).
- H. Wiedermann, *Particle Accelerator Physics*, Springer-Verlag (1995)
- J. Barnard and S. Lund, *Intense Beam Physics*, US Particle Accelerator School Notes, [http://uspas.fnal.gov/lect\\_note.html](http://uspas.fnal.gov/lect_note.html) (2006)
- F. Sacherer, *Transverse Space-Charge Effects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968).
- S. Lund and B. Bulkh, Review Article: *Stability Properties of the Transverse Envelope Equations Describing Intense Beam Transport*, PRST-Accel. and Beams 7, 024801 (2004).
- D. Nicholson, *Introduction to Plasma Theory*, Wiley (1983)
- I. Kaphanskij and V. Vladimirkij, in *Proc. Of the Int. Conf. On High Energy Accel. and Instrumentation* (CERN Scientific Info. Service, Geneva, 1959) p. 274  
Transverse Equilibrium Distributions 105

Appendix A

Self Fields of a Uniform Density Elliptical Beam in Free Space

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = \begin{cases} -\frac{\lambda}{4\pi\epsilon_0 r_x r_y} & ; \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0 & ; \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

$$\frac{\partial \phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r} \quad \text{as } r \rightarrow \infty.$$

The solution to this system to an arb. constant has been formally constructed by Landau & Lifshitz and others (gravitational field analog) as:

$$\phi = -\frac{\lambda}{4\pi\epsilon_0} \left\{ \int_0^\xi \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} + \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \left( \frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s} \right) \right\} + \text{const.}$$

where

$$\begin{cases} \xi = 0 : s & \text{when } (x/r_x)^2 + (y/r_y)^2 < 1 \\ \xi : \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} = 1 & \text{when } (x/r_x)^2 + (y/r_y)^2 > 1 \\ \text{root of } r_x^2 + \xi & r_y^2 + \xi \end{cases}$$

Trivially for  $x=y=0$

$$\phi(x=y=0) = \text{const.}$$

Calculate:

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= -\frac{\lambda}{4\pi\epsilon_0} \left\{ \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \frac{-x}{r_x^2+s} \right. \\ &\quad \left. - \frac{1}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \left[ 1 - \frac{x^2}{r_x^2+s} - \frac{y^2}{r_y^2+s} \right] \frac{\partial s}{\partial x} \right\} \end{aligned}$$

$$\text{If } \xi \neq 0 \Rightarrow \left. \left\{ 1 - \frac{x^2}{r_x^2+s} - \frac{y^2}{r_y^2+s} \right\} \right|_{\partial s / \partial x} \Rightarrow \text{2nd term vanishes}$$

$$\xi = 0 \Rightarrow \left. \left\{ \frac{\partial s}{\partial x} \right\} \right|_{\partial s / \partial x}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0} \int_{\xi}^{\infty} \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \frac{x}{r_x^2+s}$$

by symmetry

$$\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0} \int_{\xi}^{\infty} \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \frac{y}{r_y^2+s}$$

Differentiating again and using the chain rule:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\lambda}{2\pi\epsilon_0} \left\{ \int_{\xi}^{\infty} \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \left[ \frac{1}{r_x^2+s} + \frac{1}{r_y^2+s} \right] \right.$$

$$\left. - \frac{1}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \left[ \frac{x ds/dx}{r_x^2+s} + \frac{y ds/dy}{r_y^2+s} \right] \right\}$$

Must show that the r.h.s. reduces to the needed forms for:

case 1 exterior  $\xi$  satisfies:  $\frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} = 1$

case 2 interior  $\xi = 0$

case 1 (exterior:  $\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1$ )

Differentiate  $\frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} = 1$

$$\Rightarrow \frac{\partial \xi}{\partial x} = \frac{2x}{(r_x^2+\xi)} \left[ \frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2} \right]$$

$$\frac{\partial \xi}{\partial y} = \frac{2y}{(r_y^2+\xi)} \left[ \frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2} \right]$$

$$\Rightarrow \frac{x \cdot \partial \xi / \partial x}{r_x^2+\xi} + \frac{y \cdot \partial \xi / \partial y}{r_y^2+\xi} = 2 \left[ \frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2} \right] \frac{1}{\left[ \frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2} \right]} = 2$$

Also need integrals like:

$$I_x(\xi) = \int_{\xi}^{\infty} \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}} \frac{1}{r_x^2+s} = 2 \int_{\sqrt{r_x^2+\xi}}^{\infty} \frac{dw}{\sqrt{r_x^2+r_y^2+w^2}}^{1/2}$$

This integral can be done using tables:

$$I_x(\xi) = \frac{Zw}{(r_x^2 - r_y^2)\sqrt{r_x^2 - r_y^2 + w^2}} \Big|_{\substack{w \rightarrow \infty \\ w = \sqrt{r_x^2 + \xi^2}}} = \frac{Z}{r_x^2 - r_y^2} - \frac{Z\sqrt{r_y^2 + \xi^2}}{(r_x^2 - r_y^2)\sqrt{r_x^2 + \xi^2}}$$

Similarly:

$$I_y(\xi) = \int_3^\infty \frac{ds}{[(r_x^2 + s)(r_y^2 + s)]^{1/2}} \frac{1}{(r_y^2 + s)} = \frac{Z}{r_y^2 - r_x^2} - \frac{Z\sqrt{r_x^2 + \xi^2}}{(r_y^2 - r_x^2)\sqrt{r_y^2 + \xi^2}}$$

$$\int_0^\infty \frac{ds}{[(r_x^2 + s)(r_y^2 + s)]^{1/2}} \left[ \frac{1}{r_x^2 + s} + \frac{1}{r_y^2 + s} \right] = I_x(\xi) + I_y(\xi)$$

$$= \frac{Z}{r_x^2 - r_y^2} \left( \frac{\sqrt{r_x^2 + \xi^2}}{\sqrt{r_y^2 + \xi^2}} - \frac{\sqrt{r_y^2 + \xi^2}}{\sqrt{r_x^2 + \xi^2}} \right) = \frac{Z}{[(r_x^2 + \xi^2)(r_y^2 + \xi^2)]^{1/2}}$$

Using these results:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\lambda}{2\pi\epsilon_0} \left\{ \frac{Z}{[(r_x^2 + \xi^2)(r_y^2 + \xi^2)]^{1/2}} - \frac{Z}{[(r_x^2 + \xi^2)(r_y^2 + \xi^2)]^{1/2}} \right\}$$

$$= 0 \quad \text{checks.} \quad \checkmark$$

Case 2 (Interior:  $x^2/r_x^2 + y^2/r_y^2 < 1$ )

$$\xi = 0 \Rightarrow \frac{x \partial \phi / \partial x}{r_x^2 + \xi^2} + \frac{y \partial \phi / \partial y}{r_y^2 + \xi^2} = 0$$

$$\Rightarrow I_x(\xi=0) = I_y(\xi=0) = \frac{Z}{(r_x + r_y)r_x} \quad \text{and} \quad I_y(\xi=0) = \frac{Z}{(r_x + r_y)r_y}$$

Using these results:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{\lambda}{2\pi\epsilon_0} \left\{ \frac{Z}{r_x r_y} - 0 \right\} = -\frac{\lambda}{\epsilon_0 \pi r_x r_y} \quad \text{checks}$$

Finally, check the limiting form outside the beam  
for  $r$  large  $\Rightarrow \xi$  large.

$$-\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0} x I_x(\xi)$$

$$\lim_{r \rightarrow \infty} I_x(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

$$-\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0} y I_y(\xi)$$

$$\lim_{r \rightarrow \infty} I_y(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

Thus:

$$\lim_{r \rightarrow \infty} -\frac{\partial \phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0} \frac{x}{r^2} \quad \checkmark$$

$$\lim_{r \rightarrow \infty} -\frac{\partial \phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0} \frac{y}{r^2} \quad \checkmark$$

These have the correct limiting forms for a line charge at the origin. Completing the verification of the general formula.

In the beam ( $x^2/r_x^2 + y^2/r_y^2 \leq 1, z=0$ ), the formula reduces to:

$$\phi = -\frac{\lambda}{4\pi\epsilon_0} \left\{ x^2 I_x(z=0) + y^2 I_y(z=0) \right\} + \text{const.}$$

$$= -\frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{zx^2}{I_x(r_x+r_y)} + \frac{zy^2}{I_y(r_x+r_y)} \right\} + \text{const.}$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0} \left\{ \frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)} \right\} + \text{const.}}$$

The case of an axisymmetric beam with

$$r_x = r_y = r_b$$

is easy to construct explicitly and is included in the homework problems.

There is also an alternative way to do this field calculation, that is less direct but more efficient. We carry out this proof now since steps involved are useful for other purposes.

A density profile with elliptic symmetry can be expressed as:

$$n(x, y) = n\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

Here we do not assume a specific uniform density profile and we leave  $n(x^2/r_x^2 + y^2/r_y^2)$  arbitrary outside of having elliptic symmetry. The solution to the 2D Poisson equation in free-space

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{g n}{\epsilon_0}$$

is then given by

$$\phi = -\frac{g r_x r_y}{4\epsilon_0} \int_0^\infty d\xi \frac{J(2\xi)}{\sqrt{r_x^2 + \xi^2} \sqrt{r_y^2 + \xi^2}}$$

$$\mathcal{Z} \equiv \frac{x^2}{r_x^2 + \xi^2} + \frac{y^2}{r_y^2 + \xi^2}$$

where  $J(\mathcal{Z})$  is a function defined such that:

$$n(x, y) = \left.\frac{d\eta(\mathcal{Z})}{d\mathcal{Z}}\right|_{\mathcal{Z}=0}$$

This choice for  $\eta(\mathcal{Z})$  can always be made.

We first prove that this solution is valid by direct substitution:

$$\mathcal{V} = \frac{x^2}{r_x^2 + \xi} + \frac{y^2}{r_y^2 + \xi} \Rightarrow \frac{\partial \mathcal{V}}{\partial x} = \frac{2x}{r_x^2 + \xi}, \quad \frac{\partial^2 \mathcal{V}}{\partial x^2} = \frac{2}{r_x^2 + \xi}$$

$$\frac{\partial \mathcal{V}}{\partial y} = \frac{2y}{r_y^2 + \xi}, \quad \frac{\partial^2 \mathcal{V}}{\partial y^2} = \frac{2}{r_y^2 + \xi}$$

Substitute in Poisson's equation and use the chain rule and results above:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -\frac{q_{fx} q_{fy}}{4\epsilon_0} \int_0^\infty d\xi \left( \frac{d^2}{d\xi^2} \left( \frac{4x^2}{(r_x^2 + \xi)^2} + \frac{4y^2}{(r_y^2 + \xi)^2} \right) + \frac{d\eta}{d\xi} \left( \frac{2}{r_x^2 + \xi} + \frac{2}{r_y^2 + \xi} \right) \right)$$

$$\text{Note: } \frac{d\mathcal{V}}{d\xi} = - \left[ \frac{x^2}{(r_x^2 + \xi)^2} + \frac{y^2}{(r_y^2 + \xi)^2} \right] d\xi$$

so the first integral can be simplified by partial integration:

$$\begin{aligned} \int_0^\infty d\xi \left( \frac{d^2}{d\xi^2} \left( \frac{4x^2}{(r_x^2 + \xi)^2} + \frac{4y^2}{(r_y^2 + \xi)^2} \right) \right) &= -4 \int_0^\infty d\xi \frac{d^2}{d\xi^2} \frac{d\mathcal{V}}{d\xi} \\ &= -4 \int_0^\infty d\xi \frac{d}{d\xi} \left( \frac{d\mathcal{V}}{d\xi} \right) = -4 \int_0^\infty d\xi \frac{d}{d\xi} \left[ \frac{d\eta}{d\xi} \right] + 4 \int_0^\infty d\xi \frac{d\eta}{d\xi} \frac{d}{d\xi} \frac{1}{\sqrt{r_x^2 + \xi} \sqrt{r_y^2 + \xi}} \\ &= -4 \frac{d\eta}{d\xi} \Big|_{\xi=0}^{\xi=\infty} - 2 \int_0^\infty d\xi \frac{d\eta}{d\xi} \left( \frac{1}{r_x^2 + \xi} + \frac{1}{r_y^2 + \xi} \right) \\ &= \frac{4}{r_x r_y} \frac{d\eta}{d\xi} \Big|_{\xi=0}^{\infty} - 2 \int_0^\infty d\xi \frac{d\eta}{d\xi} \left( \frac{1}{r_x^2 + \xi} + \frac{1}{r_y^2 + \xi} \right) \end{aligned}$$

cancel 2nd integral

Thus:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -g f_x f_y \left. \frac{d\eta(x)}{dx} \right|_{z=0}$$

But  $\left. \frac{d\eta(x)}{dx} \right|_{z=0} = n(x, y)$  by definition.

$$\Rightarrow \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi = -g n(x, y) \quad \text{verifying the result.}$$

For a uniform density ellipse we take:

$$\eta(x) = \begin{cases} \lambda & ; x < 1 \\ g f_x f_y & ; 1 \leq x \leq 1 \\ 1 & ; x > 1 \end{cases} \Rightarrow \frac{d\eta(x)}{dx} = \begin{cases} \frac{\lambda}{g f_x f_y} & ; x < 1 \\ 0 & ; x > 1 \end{cases}$$

Thus

$$\left. \frac{d\eta(x)}{dx} \right|_{z=0} = \begin{cases} \frac{\lambda}{g f_x f_y} & ; x \leq 1 \\ 0 & ; x \geq 1 \end{cases} = \begin{cases} \frac{\lambda}{g f_x f_y} & ; \frac{x^2}{f_x^2} + \frac{y^2}{f_y^2} \leq 1 \\ 0 & ; \frac{x^2}{f_x^2} + \frac{y^2}{f_y^2} > 1 \end{cases}$$

$$\therefore \left. \frac{d\eta(x)}{dx} \right|_{z=0} = n(x, y) \quad \text{for a uniform density elliptical beam. with radii}$$

Apply these results to calculate  $\phi$  interior to a uniform density elliptical beam. with radii  $f_x, f_y$  and density  $\lambda(g f_x f_y)$

$$\phi = -g f_x f_y \int_0^\infty d\zeta \frac{d\eta(x)}{\sqrt{f_x^2 + \zeta^2} \sqrt{f_y^2 + \zeta^2}}$$

$$x = \frac{x^2}{f_x^2 + \zeta^2} + \frac{y^2}{f_y^2 + \zeta^2} \quad \text{if } \frac{x^2}{f_x^2} + \frac{y^2}{f_y^2} \leq 1 \rightarrow \text{then}$$

$$x^2 + y^2 \leq f_x^2 + f_y^2 \quad \text{for all}$$

Using this and the result above,  $0 \leq \zeta < \infty$   
for  $\eta(x)$ ,  $\phi$  inside the elliptical beam is:

$$\phi = -g f_x f_y \int_0^\infty d\zeta \frac{\lambda}{g f_x f_y} \left[ \frac{x^2}{(f_x^2 + \zeta^2)^{3/2} (f_y^2 + \zeta^2)^{1/2}} + \frac{y^2}{(f_x^2 + \zeta^2)^{1/2} (f_y^2 + \zeta^2)^{3/2}} \right]$$

$$\phi = -\lambda \left\{ \frac{x^2}{4\pi\epsilon_0} \int_0^\infty \frac{ds}{(r_x^2+s)^{3/2}} \frac{1}{(r_y^2+s)^{1/2}} + \frac{y^2}{4\pi\epsilon_0} \int_0^\infty \frac{ds}{(r_y^2+s)^{3/2}} \frac{1}{(r_x^2+s)^{1/2}} \right\}$$

Using Mathematica or Integral tables:

$$\int_0^\infty \frac{ds}{(r_x^2+s)^{3/2}} \frac{1}{(r_y^2+s)^{1/2}} = \frac{z}{r_x(r_x+r_y)}$$

$$\int_0^\infty \frac{ds}{(r_y^2+s)^{3/2}} \frac{1}{(r_x^2+s)^{1/2}} = \frac{z}{r_y(r_x+r_y)}$$

Hence

$$\phi = -\lambda \left\{ \frac{x^2}{2\pi\epsilon_0 r_x(r_x+r_y)} + \frac{y^2}{2\pi\epsilon_0 r_y(r_x+r_y)} \right\} + \text{const}$$

since an overall constant can always be added to  $\phi$   
 (The integral has a reference choice  $\phi(x=y=0) = 0$  built in.).

The steps introduced in this proof can also be used to show that:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_1 = -\lambda \frac{r_x}{4\pi\epsilon_0 r_x+r_y}$$

$$\lambda = g \int d^2x n$$

$$\langle y \frac{\partial \phi}{\partial y} \rangle_1 = -\lambda \frac{r_y}{4\pi\epsilon_0 r_x+r_y}$$

$$r_x = 2\langle x^2 \rangle^{1/2}$$

$$r_y = 2\langle y^2 \rangle^{1/2}$$

for any elliptic symmetry density profile  
 $n(x,y) = n(x^2/r_x^2 + y^2/r_y^2)$ . In the intro. lectures, these results were employed to show that the kV envelope equations with evolving emittances can be applied to elliptic symmetry beams. This result was first demonstrated by Sacherer: [IEEE Trans Nucl. Sci. 18, 1105 (1971)]

## Canonical Transformation of the ICV Distribution

The single-particle equations of motion can be derived from the Hamiltonian:  $(\frac{d\vec{x}_1}{ds} = \frac{\partial H}{\partial \vec{p}_1}, \frac{d\vec{x}'_1}{ds} = -\frac{\partial H}{\partial \vec{x}_1})$

$$H_1(x, y, x', y', s) = \frac{1}{2} x'^2 + \left[ P_x(s) - \frac{ZQ}{f_x(s)[f_x(s) + f_y(s)]} \right] \frac{x^2}{Z}$$

$$+ \frac{1}{2} y'^2 + \left[ P_y(s) - \frac{ZQ}{f_y(s)[f_x(s) + f_y(s)]} \right] \frac{y^2}{Z}$$

Perform a canonical transform to new variables

$X, Y, X', Y'$  using the generating function

$$F_2(x, y, X, Y) = \frac{x}{w_x} \left[ X' + \frac{x w_x'}{Z} \right] + \frac{y}{w_y} \left[ Y' + \frac{y w_y'}{Z} \right]$$

Then:

$$\boxed{X = \frac{\partial F_2}{\partial X'} = \frac{x}{w_x}}$$

$$\boxed{Y = \frac{\partial F_2}{\partial Y'} = \frac{y}{w_y}}$$

Ref:

R.C. Davidson,  
"Physics of Nonneutral Plasmas"  
Addison-Wesley, 1990

comment!

$$x' = \frac{\partial F_2}{\partial X} = \frac{1}{w_x} (X' + x w_x')$$

$$y' = \frac{\partial F_2}{\partial Y} = \frac{1}{w_y} (Y' + y w_y')$$

and solving for  $X', Y'$ :

$$\boxed{X' = w_x x' - x w_x'}$$

$$\boxed{Y' = w_y y' - y w_y'}$$

Here,  $\underline{X}' \neq \frac{d\underline{X}}{ds}$ ,  $\underline{X}'$  merely denotes the conjugate variable to  $\underline{X}$

Also,  $\underline{X}, \underline{X}'$  both have dim.  
meters<sup>1/2</sup>

The Courant-Snyder Invariants are then simply expressed:

$$\boxed{c_1 = \underline{X}^2 + \underline{X}'^2 = \text{const}}$$

$$\boxed{c_2 = \underline{Y}^2 + \underline{Y}'^2 = \text{const}}$$

One can show from the transformations that:

$$dx dy = w_x w_y d\bar{x} d\bar{y}$$

$$dx' dy' = \frac{d\bar{x}' d\bar{y}'}{w_x w_y}$$

$$dx dy dx' dy' = d\bar{x} d\bar{y} d\bar{x}' d\bar{y}' *$$

\* Property  
of canonical  
transforms  
in general.  
Results from  
structure of  
Generating Function

Therefore, the distribution in transformed phase space variables is the same as for the original variables:

$$f_i(\bar{x}, \bar{y}, \bar{x}', \bar{y}', s) = f_i(x, y, x', y', s)$$

$$= \frac{\lambda}{\epsilon \pi^2 \epsilon_x \epsilon_y} \delta \left[ \frac{\bar{x}^2 + \bar{x}'^2}{\epsilon_x} + \frac{\bar{y}^2 + \bar{y}'^2}{\epsilon_y} - 1 \right]$$

Now examine the density:

$$n(x, y) = \int dx' dy' f_i = \int \frac{d\bar{x}' d\bar{y}'}{w_x w_y} f_i$$

$$U_x = \bar{x}' / \sqrt{\epsilon_x}, \quad U_y = \bar{y}' / \sqrt{\epsilon_y}$$

$$r_x = \sqrt{\epsilon_x} w_x, \quad r_y = \sqrt{\epsilon_y} w_y \quad dU_x dU_y = \frac{d\bar{x}' d\bar{y}'}{\sqrt{\epsilon_x \epsilon_y}}$$

$$n = \frac{\lambda}{\epsilon \pi^2 r_x r_y} \int dU_x dU_y \delta \left[ U_x^2 + U_y^2 - \left( 1 - \frac{\bar{x}^2}{\epsilon_x} - \frac{\bar{y}^2}{\epsilon_y} \right) \right]$$

Exploit the cylindrical symmetry:

$$U_1^2 = U_x^2 + U_y^2$$

$$dU_x dU_y = d\psi dU_1 dU_1 = d\psi \frac{dU_1^2}{2}$$

$$n(x, y) = \frac{\lambda}{g\pi r_x r_y} \int_0^{2\pi} d\psi \int_0^\infty \frac{dU_1^2}{2} \delta\left(U_1^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right)$$

Thus:

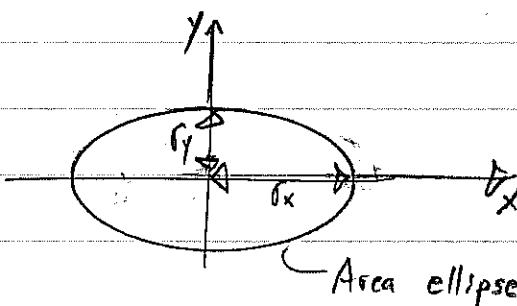
$$n(x, y) = \frac{\lambda}{g\pi r_x r_y} \int_0^\infty dU_1^2 \delta\left(U_1^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right)$$

$$= \begin{cases} \frac{\lambda}{g\pi r_x r_y} = \hat{n} & ; \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \leq 1 \\ 0 & ; \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

Showing that the singular KV distribution yields the required uniform density beam of elliptical cross-section.

Note

$$\hat{n} = \frac{\lambda}{g\pi r_x r_y}$$



$$= \pi r_x r_y$$

$$\lambda = g \hat{n} \pi r_x r_y$$

for uniform density.

S. M. Lund BY

An interesting footnote to this appendix is that an identity of generating functions can be used to transform the KV distribution in standard quadratic form:

$$f_C \sim S [X'^2 + Y'^2 + X^2 + Y^2 - \text{const}]$$

to other sets of variables. This will generate other distributions with KV form for skew coupling and other effects. It would not be logical to label such distributions as "new" as has been done in the literature. However, identifying physically relevant transforms has practical value.

Inverse transform:

$$x = w_x \bar{x}$$

$$w_x x' = \bar{x}' + x w_x' \Rightarrow x' = \frac{\bar{x}' + w_x' \bar{x}}{w_x}$$

$$x = w_x \bar{x}$$

$$x' = \bar{x}' / w_x + w_x' \bar{x}$$

$$y = w_y \bar{y}$$

$$y' = \bar{y}' / w_y + w_y' \bar{y}$$

Next,

$$\frac{d}{ds} \bar{x} = \frac{x'}{w_x} - \frac{x}{w_x^2} w_x'$$

$$= \frac{\bar{x}'}{w_x^2} + \frac{w_x' \bar{x}}{w_x} - \frac{w_x' \bar{x}}{w_x} = \frac{\bar{x}'}{w_x^2}$$

Thus,

$$\frac{d}{ds} \bar{x} = \frac{\bar{x}'}{w_x^2}$$

$$\frac{d}{ds} \bar{y} = \frac{\bar{y}'}{w_y^2}$$

$$\frac{d}{ds} \bar{x}' = \cancel{w_x' \bar{x} + w_x x''} - \cancel{x' w_x + x w_x''}$$

$$\Rightarrow x'' = \frac{\frac{d}{ds} \bar{x}'}{w_x} + w_x'' \bar{x}$$

$$y'' = \frac{\frac{d}{ds} \bar{y}'}{w_y} + w_y'' \bar{y}$$

Apply in Egn of motion:

$$\ddot{x}'' + R_x \dot{x} - \frac{ZQx}{(F_x + F_y) F_x} = 0$$

$$\frac{\frac{d}{ds} \ddot{X}'}{W_x} + W_x'' \ddot{X} - R_x W_x \ddot{X} - \frac{ZQ W_x \ddot{X}}{(F_x + F_y) F_x} = 0$$

$\underbrace{\quad}_{\frac{1}{W_x^2} \text{ from } W_x \text{ egn}}$

$$\frac{\frac{d}{ds} \ddot{X}'}{(F_x + F_y) F_x} + N \left( W_x'' + R_x W_x - \frac{ZQ W_x}{(F_x + F_y) F_x} \right) \ddot{X} = 0$$

$\frac{d}{ds} \ddot{X}' + \frac{1}{W_x^2} \ddot{X} = 0 \quad , \quad \frac{d}{ds} \ddot{X} = \frac{\ddot{X}'}{W_x^2}$	$\frac{d}{ds} \ddot{Y}' + \frac{1}{W_y^2} \ddot{Y} = 0 \quad , \quad \frac{d}{ds} \ddot{Y} = \frac{\ddot{Y}'}{W_y^2}$
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Eqn. 1

Following Davidson, these eqns can be solved

using the method of variation of parameters

$$X(s) = X_0 \cos \psi_x(s) + X_p' \sin \psi_x(s)$$

$$\psi_x(s) = \int_{s_1}^s \frac{ds'}{W_x^2(s')} \quad ; \quad \psi'_x(s) = \frac{1}{W_x^2(s)}$$

This also demonstrates explicitly the  
C.S. Invariant

$$\dot{X}^2 + \dot{X}'^2 = \text{const.}$$

Note:

$$\frac{d}{ds} \ddot{X} = \frac{\ddot{X}'}{W_x^2}$$

Transformed Hamiltonian:

$$\frac{d}{ds} \underline{X} - \frac{\partial \tilde{H}}{\partial \underline{X}'} = \frac{\underline{X}'}{w_x^2}$$

$$\frac{d}{ds} \underline{Y} = \frac{\partial \tilde{H}}{\partial \underline{Y}'} = \frac{\underline{Y}'}{w_y^2}$$

$$\frac{d}{ds} \underline{X}' = -\frac{\partial \tilde{H}}{\partial \underline{X}} = -\frac{1}{w_x^2} \underline{X}$$

$$\frac{d}{ds} \underline{Y}' = -\frac{\partial \tilde{H}}{\partial \underline{Y}} = -\frac{1}{w_y^2} \underline{Y}$$

$$\tilde{H} = \tilde{H}(\underline{X}, \underline{Y}, \underline{X}', \underline{Y}')$$

Transformed Hamiltonian

$$\Rightarrow \tilde{H} = \frac{1}{2w_x^2} \underline{X}'^2 + \frac{1}{2w_y^2} \underline{Y}'^2 + \frac{1}{2w_x^2} \underline{X}^2 + \frac{1}{2w_y^2} \underline{Y}^2 + \text{const.}$$

Note that  $\tilde{H}$  is still explicitly  $s$ -dependent based on  $w_x$  and  $w_y$  lattice functions.

This Hamiltonian can also be found using results from the theory of canonical transformations.

$$\Rightarrow \tilde{H} = H + \frac{\partial F_2}{\partial s}$$

and the coordinate transform expressions.